5 Analysis of an interacting particle system with jumps

In the previous section we used a coupling to prove propagation of chaos, in this section the proof is based on tightness and uniqueness of the limiting distribution.

We focus on the toy model of interacting neurons introduced in Section 1.3. This section is based on the articles of De Masi et al. [DMGLP15] and of Fournier and Löcherbach [FL16]. Let us recall the interacting particle system and its nonlinear asymptotic.
Let us consider a family of neurons of size N. We denote by $X_t^{i,N}$ the membrane potential of neuron i at time $t \geqslant 0$.
We assume that initially the potentials are i.i.d: $(X_0^i)_{i\geqslant 0}$ is a sequence of nonnegative i.i.d initial potentials, with distribution g_0 on $\mathbb{R}_+$.
We introduce the two parameters of the model:

- the spiking rate of the system is modeled by the function $f : \mathbb{R}_+ \to \mathbb{R}_+$;
- $\lambda \geqslant 0$ the attraction force to the mean potential.

The evolution in time of the potentials of each neuron is given by the following SDEs:

$$\begin{aligned} X_t^{i,N} =& X_0^i - \lambda \int_0^t \left(X_s^{i,N} - \overline{X}_s^N \right) \mathrm{d}s \\ & - \int_0^t \int_{\mathbb{R}_+} X_s^{i,N} \mathbb{1}_{\left\{ z \leqslant f(X_{s-}^{i,N}) \right\}} Q^i(\mathrm{d}s, \mathrm{d}z) + \frac{1}{N} \sum_{j \neq i} \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\left\{ z \leqslant f(X_{s-}^{j,N}) \right\}} Q^j(\mathrm{d}s, \mathrm{d}z), \end{aligned} \tag{5.1}$$

with $\left(Q^i(\mathrm{d}s, \mathrm{d}z)\right)_{i\geqslant N}$ i.i.d. Poisson measures on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\mathrm{d}s\mathrm{d}z$, independent of $(X_0^i)_{i\geqslant 0}$, and $\overline{X}_t^N = \frac{1}{N}\sum_{i=1}^N X_t^{i,N}$ is the empirical average potential of the system.
The first integral models the attraction to the mean value, the second integral models the jump to 0 of the potential when neuron i spikes, and finally the last integral models the additive potential of size $\frac{1}{N}$ at each spike of another neuron.

Strong well-posedness of the system (5.1) holds under the following assumption on the jump rate, see [FL16, Proposition 2].

Assumption 2. *f is a non-decreasing function of class $\mathcal{C}^1$, with $f(0) = 0$, $f(x) > 0$ for $x > 0$, and* $\lim_{x\to\infty} f(x) = \infty$.

Since f is non decreasing, the higher the potential, the bigger the probability of spiking.
The associated nonlinear SDE is the following

$$X_t = X_0 - \lambda \int_0^t (X_s - \mathbb{E}[X_s])\mathrm{d}s - \int_0^t \int_{\mathbb{R}_+} X_{s-} \mathbb{1}_{\{z\leqslant f(X_{s-})\}} Q(\mathrm{d}s, \mathrm{d}z) + \int_0^t \mathbb{E}[f(X_s)]\mathrm{d}s, \tag{5.2}$$

where $Q(\mathrm{d}s, \mathrm{d}z)$ is a Poisson measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $\mathrm{d}s\mathrm{d}z$, and X_0 is independent of Q, with distribution g_0. When g_0 is compactly supported, there is uniqueness of the solution to (5.2) satisfying $\int_0^t \mathbb{E}[X_s f(X_s)]\mathrm{d}s < \infty$ for any $t \geqslant 0$, under Assumption 2 (see [FL16, Theorem 4]).
We easily note that for any $t \geqslant 0$, $X_t^{i,N} \geqslant 0$ and $X_t \geqslant 0$, and that they are càdlàg functions (in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$).

We prensent here the proof of [FL16, Theorem 5] on the propagation of chaos for the system based on tightness arguments.

Theorem 5.1. *Assume that the initial potentials are integrable, i.e.* $\int x g_0 \mathrm{d}x < \infty$. *Then,*

1. *The sequence of empirical measures* $\bar{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ *is tight in* $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.
2. *If the initial distribution is compactly supported, there is propagation of chaos to the distribution of the solution to* (5.2).

The assumption to obtain propagation of chaos can be relaxed as explained in [FL16]. In addition, under stronger condition on the jump rate f, using a coupling method and a well adapted distance, [FL16, Theorem 7] gives a speed of convergence to the chaos. They also obtain the invariant measures (when $t \to \infty$) of the nonlinear SDE (5.2) under additional assumptions in [FL16, Theorem 8]: there are two possible equilibrium, either $g(\mathrm{d}x) = \delta_0$ or

$$g(\mathrm{d}x) = \frac{p}{p+\lambda m - \lambda x} \exp\left(-\int_0^x \frac{f(y)}{p+\lambda(m-y)}\mathrm{d}y\right)\mathbb{1}_{0\leqslant x\leqslant m+p/\lambda}\mathrm{d}x \quad \text{when } \lambda > 0,$$
$$g(\mathrm{d}x) = \exp\left(-\frac{1}{p}\int_0^x f(y)\mathrm{d}y\right)\mathrm{d}x \quad \text{when } \lambda = 0,$$

where $p, m > 0$ are constants determined by the constraints $\int g(\mathrm{d}x) = 1$ and $\int x g(\mathrm{d}x) = m$.

5.1 Tightness

By exchangeability of the system and by Proposition 2.8, the tightness of $\left(\bar{\mu}^N\right)_{N\geqslant 1}$ is equivalent to the tightness of $\left(\mathrm{Law}(X^{1,N})\right)_{N\geqslant 1}$.
Since the trajectories of $X^{1,N}$ are in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$, we use the Aldous criterion to prove tightness of $\left(\mathrm{Law}(X^{1,N})\right)_{N\geqslant 1}$ (see Theorem 3.11). We have to prove

(i) for all $T > 0$, $\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\leq T} |X_t^{1,N}| \geqslant a\right) = 0$;

(ii) for $T > 0$ and for all $a > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{N} \sup_{\substack{S,S'\text{stopping times:}\\ S\leqslant S'\leqslant S+\delta\leqslant T}} \mathbb{P}\Big(|X_{S'}^{1,N} - X_S^{1,N}| \geqslant a\Big) = 0.$$

To prove Condition (i), we control $X_t^{1,N}$ with its initial value $X_0^{1,N}$ and the initial mean value $\overline{X}_0^N$. Recall that $X_t^{1,N} \geqslant 0$ for any $t \geqslant 0$.

■ **Exercise 5.** The aim of this exercise is to prove that for any $t \geqslant 0$,

$$X_t^{i,N} \leqslant X_0^{i,N} + (4\lambda t + 4)\Big(\overline{X}_0^N + Z_t^N\Big), \tag{5.3}$$

with $Z_t^N = \frac{1}{N}\sum_{i=1}^N \int_0^t \int_{\mathbb{R}_+} \mathbb{1}_{\{z\leqslant f(2)\}} Q^i(\mathrm{d}s, \mathrm{d}z)$.

1. Prove that $x - 1 \geqslant \frac{x+1}{3} - \frac{4}{3}\mathbb{1}_{x\leqslant 2}$ for all $x \geqslant 0$.
2. From (5.1), give the equation satisfied by $\overline{X}_t^N = \frac{1}{N}\sum_{i=1}^N X_t^{i,N}$.

3. Using the two first questions, prove that

$$\frac{1}{N}\sum_{i=1}^{N}\int_0^t\int_{\mathbb{R}_+}\left(X_{s-}^{i,N}+1\right)\mathbb{1}_{\left\{z\leqslant f(X_{s-}^{i,N})\right\}}Q^i(\mathrm{d}s,\mathrm{d}z)\leqslant 3\overline{X}_0^N+4Z_t^N. \tag{5.4}$$

4. Deduce $\overline{X}_t^N \leqslant 4\overline{X}_0^N + 4Z_t^N$.
5. Conclude to obtain (5.3).

From the above estimate, we have for any $T>0$,

$$\sup_{t\leqslant T} X_t^{i,N}\leqslant X_0^{i,N}+(4\lambda T+4)\left(\overline{X}_0^N+Z_T^N\right),$$

with $Z_T^N = \frac{1}{N}\sum_{i=1}^N\int_0^T\int_{\mathbb{R}_+}\mathbb{1}_{\{z\leqslant f(2)\}}Q^i(\mathrm{d}s,\mathrm{d}z)$ the empirical mean of N i.i.d. Poisson($f(2)T$) random variables. Consequently,

$$\sup_{N\geqslant 1}\mathbb{E}[\sup_{t\leqslant T}X_t^{i,N}]\leqslant \mathbb{E}[X_0]+(4\lambda T+4)(\mathbb{E}[X_0]+f(2)T)<\infty.$$

Using Markov inequality, we easily deduce Condition (i) of Aldous' criterion.

Let us now study Condition (ii). Let $T>0$ and $\delta>0$.
Remark: Note that, in this proof, C_T denotes a generic constant arising in the tightness argument, whose value may change from one line to another.

Taking the expectation in (5.4) and using the exchangeability, we have for any $N\geqslant 1$ and $t\in[0,T]$,

$$\int_0^t \mathbb{E}\left[X_s^{1,N}f\left(X_s^{1,N}\right)\right]\mathrm{d}s\leqslant 3\mathbb{E}[X_0]+4f(2)t\leqslant C_T, \tag{5.5}$$

with $C_T = 3\mathbb{E}[X_0]+4f(2)T$.
We consider S,S' two $(\mathcal{F}_t)_{t\geqslant 0}$-stopping times, with $S\leqslant S'\leqslant S+\delta\leqslant T$. We have

$$\begin{aligned}\left|X_{S'}^{1,N}-X_S^{1,N}\right| &\leqslant \int_S^{S'}\int_{\mathbb{R}_+}X_u^{1,N}\mathbb{1}_{\left\{z\leqslant f(X_{u-}^{1,N})\right\}}Q^1(\mathrm{d}u,\mathrm{d}z)+\frac{1}{N}\sum_{i=2}^N\int_S^{S'}\int_{\mathbb{R}_+}\mathbb{1}_{\left\{z\leqslant f(X_{u-}^{i,N})\right\}}Q^i(\mathrm{d}u,\mathrm{d}z)\\ &\quad+\lambda\int_S^{S'}X_u^{1,N}\mathrm{d}u+\lambda\int_S^{S'}\overline{X}_u^N\mathrm{d}u\\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

First note that, if I_1 is positive then there was at least one jump the $X^{1,N}$, and therefore

$$\begin{aligned}\mathbb{P}(I_1>0)&\leqslant \mathbb{P}\left(\int_S^{S'}\int_{\mathbb{R}_+}\mathbb{1}_{\left\{z\leqslant f(X_{u-}^{1,N})\right\}}Q^i(\mathrm{d}u,\mathrm{d}z)\geqslant 1\right)\\ &\leqslant \mathbb{E}\left[\int_S^{S+\delta}\int_{\mathbb{R}_+}\mathbb{1}_{\left\{z\leqslant f(X_{u-}^{1,N})\right\}}Q^i(\mathrm{d}u,\mathrm{d}z)\right]=\mathbb{E}\left[\int_S^{S+\delta}f(X_u^{1,N})\mathrm{d}u\right],\end{aligned}$$

where the Markov inequality has been used in the second inequality. Since f is non-decreasing, it satisfies for any $m > 0$ and $x \geqslant 0$, $f(x) \leqslant f(m) + \frac{x}{m} f(x)$, and thus

$$\mathbb{P}(I_1 > 0) \leqslant \delta f(m) + \frac{1}{m} \int_0^T \mathbb{E}\big[X_u^{1,N} f(X_u^{1,N})\big] \mathrm{d}u \leqslant \delta f(m) + \frac{C_T}{m} = \sqrt{\delta} + \frac{C_T}{f^{-1}(\delta^{-1/2})},$$

by (5.5) and for $m = f^{-1}(\delta^{-1/2})$, where f^{-1} is the generalized inverse function: $f^{-1}(y) = \inf\{x \geqslant 0 : f(x) \geqslant y\}$.
Similarly, by exchangeability,

$$\mathbb{E}[I_2] \leqslant \frac{N-1}{N} \mathbb{E}\left[\int_S^{S+\delta} f(X_u^{1,N})\right] \mathrm{d}u \leqslant \sqrt{\delta} + \frac{C_T}{f^{-1}(\delta^{-1/2})}.$$

For the two next terms, we use that for $m > 0$ and $x \geqslant 0$, $x \leqslant m + x\frac{f(x)}{f(m)}$, and then

$$\mathbb{E}[I_3] \leqslant \lambda m \delta + \frac{\lambda}{f(m)} \int_0^T \mathbb{E}\big[X_u^{1,N} f\big(X_u^{1,N}\big)\big] \mathrm{d}u \leqslant \lambda m \delta + \frac{\lambda C_T}{f(m)} = \lambda\sqrt{\delta} + \frac{\lambda C_T}{f(\delta^{-1/2})},$$

by (5.5) and for $m = \delta^{-1/2}$. Similarly,

$$\mathbb{E}[I_3] \leqslant \lambda\sqrt{\delta} + \frac{\lambda C_T}{f(\delta^{-1/2})}.$$

Let $a > 0$, we have

$$\begin{aligned}\mathbb{P}\Big(\Big|X_{S'}^{1,N} - X_S^{1,N}\Big| \geqslant a\Big) &\leqslant \mathbb{P}\left(\bigcup_{i=1}^{4} I_i \geqslant \frac{a}{4}\right) \leqslant \mathbb{P}(I_1 > 0) + \mathbb{P}(I_2 > a/4) + \mathbb{P}(I_3 > a/4) + \mathbb{P}(I_4 > a/4) \\ &\leqslant \sqrt{\delta} + \frac{C_T}{f^{-1}(\delta^{-1/2})} + \frac{4}{a}\left((1+2\lambda)\sqrt{\delta} + \frac{C_T}{f^{-1}(\delta^{-1/2})} + \frac{2\lambda C_T}{f(\delta^{-1/2})}\right),\end{aligned}$$

by the Markov inequality. The upper bound is uniform in N, S, S', and goes to 0 when $\delta \to 0$ because $\lim_{x\to\infty} f(x) = \infty$ by assumption. Consequently, Condition (ii) is satisfied, and thus $(X^{1,N})_{N\geqslant 1}$ is tight in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$.

5.2 Propagation of chaos

By Theorem 3.6, since there is uniqueness of the solution to nonlinear SDE (5.2) and the sequence $\bar{\mu}^N$ is tight, we only have to prove the consistency: the limiting distributions of $\bar{\mu}^N$ are solution of the martingale problem associated to (5.2) (see [JS03, Theorems II.2.42 and III.2.26]).
Let g_0 be the initial distribution. We denote by Z the canonical process on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$: $Z_t(\omega) = \omega_t$. For $t \geqslant 0$, define the projection $p_t : \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+) \to \mathbb{R}_+$ by $p_t(z) = z_t$. Assume that $\bar{\mu}^N$ converges in law to μ in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$ (up to a subsequence). We want to show that

(a) $\mu \circ p_0^{-1} = g_0$;

(b) $\forall t \geqslant 0$, $\mathbb{E}_\mu\left[\int_0^t Z_s f(Z_s) \mathrm{d}x\right] < \infty$, where $\mathbb{E}_\mu$ is the expectation under the distribution μ;

(c) for all $\varphi \in \mathcal{C}_b^2(\mathbb{R}_+, \mathbb{R})$, i.e. a bounded function of class $\mathcal{C}^2$ with bounded derivatives,

$$\varphi(Z_t) - \varphi(Z_0) - \int_0^t (\varphi(0) - \varphi(Z_u))f(Z_s)\mathrm{d}s - \int_0^t \varphi'(Z_s)\Big[\mathbb{E}_\mu[f(Z_s)] + \lambda(\mathbb{E}_\mu[Z_s] - Z_s)\Big]\mathrm{d}s$$

is a μ-martingale.

Note that [FL16] proved that $\mu(Z_t - Z_{t-} \neq 0) = 0$ for all $t \geqslant 0$ (see Step 2 in the proof of Theorem 5, Section 4).

First, we observe that $\mu \circ \pi_0^{-1}$ is the limit in law of $\bar{\mu}^N \circ \pi_0^{-1} = \frac{1}{N}\sum_{i=1}^N \delta_{X_0^{i,N}}$. Since $X_0^{i,N}$ are i.i.d. random variables with distribution g_0, by the Law of Large Numbers we deduce that $\mu \circ \pi_0^{-1} = g_0$. Second, for $t \geqslant 0$ and $K > 0$, by Fatou Lemma, and since $x \mapsto xf(x) \wedge K$ is a bounded function,

$$\begin{aligned}\mathbb{E}_\mu\left[\int_0^t (Z_s f(Z_s) \wedge K)\mathrm{d}s\right] &\leqslant \liminf_N \mathbb{E}_{\bar{\mu}^N}\left[\int_0^t (Z_s f(Z_s) \wedge K)\mathrm{d}s\right] \\ &= \liminf_N \frac{1}{N}\sum_{i=1}^N \mathbb{E}\left[\int_0^t (X_s^{i,N} f(X_s^{i,N}) \wedge K)\mathrm{d}s\right] \leqslant C_t < \infty\end{aligned}$$

by (5.5), with C_t the constant introduced in Section 5.1 independent of K . The conclusion follows by letting $K \to \infty$ and using the monotone convergence theorem.
Finally, we prove Point (c). To this end, let $k \geqslant 1$, $0 \leqslant s_1 < \ldots < s_k < s < t$, $\varphi_1, \ldots, \varphi_k \in \mathcal{C}_b(\mathbb{R}_+)$, any $\varphi \in \mathcal{C}_b^2(\mathbb{R}_+)$, and define

$$\begin{aligned}F(\mu) := \mathbb{E}_\mu\Bigg[&\varphi_1(Z_{s_1})\ldots\varphi_1(Z_{s_k})\times \\ &\left(\varphi(Z_t) - \varphi(Z_s) - \int_s^t f(Z_u)(\varphi(0) - \varphi(Z_u))\mathrm{d}u - \int_s^t \varphi'(Z_u)\Big(\mathbb{E}_\mu[f(Z_u)] + \lambda(\mathbb{E}_\mu[Z_u] - Z_u)\Big)\mathrm{d}u\right)\Bigg].\end{aligned}$$

We want to show that $F(\mu) = \lim_{N\to\infty} \mathbb{E}[F(\bar{\mu}^N)] = 0$. [FL16] proves the continuity of the function F, ensuring that $F(\mu) = \lim_{N\to\infty} \mathbb{E}[F(\bar{\mu}^N)]$. We only detail here the proof of the convergence $\lim_{N\to\infty} \mathbb{E}[F(\bar{\mu}^N)] = 0$ to conclude $F(\mu) = 0$ and therefore μ is a solution of the martingale problem associated to (5.2).
The remainder of the proof uses Itô's formula for jump processes. If you are not familiar with this formula, you can stop reading here.

Remark: C_F denotes a generic constant arising in the consistency argument, whose value may change from one line to another.

We have

$$\begin{aligned}F(\bar{\mu}^N) = \frac{1}{N}\sum_{i=1}^N &\varphi_1(X_{s_1}^{i,N})\ldots\varphi_1(X_{s_k}^{i,N})\times \\ &\Bigg[\varphi(X_t^{i,N}) - \varphi(X_s^{i,N}) - \int_s^t f(X_u^{i,N})\big(\varphi(0) - \varphi(X_u^{i,N})\big)\mathrm{d}u - \lambda\int_s^t \varphi'(X_u^{i,N})(\overline{X}_u^N - X_u^{i,N})\mathrm{d}u \\ &\qquad - \frac{1}{N}\sum_{j=1}^N \int_s^t \varphi'(X_u^{i,N})f(X_u^{j,N})\mathrm{d}u\Bigg].\end{aligned}$$

Using Itô formula for the system (5.1) of jump processes, we have

$$
\begin{aligned}
&\varphi(X_t^{i,N})\\
&=\varphi(X_0^{i,N})+\int_0^t\int_{\mathbb{R}_+}\Big(\varphi(0)-\varphi(X_{u-}^{i,N})\Big)\mathbb{1}_{\big\{z\leqslant f(X_{u-}^{i,N})\big\}}Q^i(\mathrm{d}u,\mathrm{d}z)+\lambda\int_0^t\varphi'(X_{u-}^{i,N})(\overline{X}_u^N-X_u^{i,N})\mathrm{d}u\\
&+\sum_{j\neq i}\int_0^t\int_{\mathbb{R}_+}\bigg(\varphi\bigg(X_{u-}^{j,N}+\frac{1}{N}\bigg)-\varphi(X_{u-}^{j,N})\bigg)\mathbb{1}_{\big\{z\leqslant f(X_{u-}^{j,N})\big\}}Q^j(\mathrm{d}u,\mathrm{d}z).
\end{aligned}
$$

Introducing the compensator $\widetilde{Q}^i(\mathrm{d}u,\mathrm{d}z)=Q^i(\mathrm{d}u,\mathrm{d}z)-\mathrm{d}u\mathrm{d}z$, defining

$$
\begin{aligned}
M_t^{i,N}&=\int_0^t\int_{\mathbb{R}_+}\Big(\varphi(0)-\varphi(X_{u-}^{i,N})\Big)\mathbb{1}_{\big\{z\leqslant f(X_{u-}^{i,N})\big\}}\widetilde{Q}^i(\mathrm{d}u,\mathrm{d}z)\\
\Delta_t^{i,N}&=\sum_{j\neq i}\int_0^t\int_{\mathbb{R}_+}\bigg(\varphi\bigg(X_{u-}^{j,N}+\frac{1}{N}\bigg)-\varphi(X_{u-}^{j,N})\bigg)\mathbb{1}_{\big\{z\leqslant f(X_{u-}^{j,N})\big\}}Q^j(\mathrm{d}u,\mathrm{d}z)\\
&\quad-\int_0^t\varphi'(X_u^{i,N})\frac{1}{N}\sum_{j=1}^N f(X_u^{j,N})\mathrm{d}u,
\end{aligned}
$$

we observe that

$$
F(\bar{\mu}^N)=\frac{1}{N}\sum_{i=1}^N\varphi_1(X_{s_1}^{i,N})\ldots\varphi_1(X_{s_k}^{i,N})\Big[\Big(M_t^{i,N}-M_s^{i,N}\Big)+\Big(\Delta_t^{i,N}-\Delta_s^{i,N}\Big)\Big].
$$

It is known that $(M^{i,N})$ are $(\mathcal{F}_t)_{t\geqslant 0}$-martingales. In addition, the Poisson measures $(Q^i)_{i\geqslant 1}$ being independent, the martingales $M^{i,N}$ are orthogonal. Using exchangeability, and the boundedness of φ_k, there exist a constant $C_F>0$ such that

$$
\mathbb{E}[\big|F(\bar{\mu}^N)\big|]\leqslant\frac{C_F}{\sqrt{N}}\mathbb{E}\Big[\Big(M_t^{1,N}-M_t^{1,N}\Big)^2\Big]^{1/2}+C_F\,\mathbb{E}\Big[\Big|\Delta_t^{1,N}\Big|+\big|\Delta_s^{1,N}\big|\Big]
$$

As φ is bounded, $f(x)\leqslant f(1)+xf(x)$, and (5.5), we also have

$$
\mathbb{E}\Big[\Big(M_t^{1,N}-M_t^{1,N}\Big)^2\Big]=\int_s^t\mathbb{E}\Big[\big(\varphi(0)-\varphi(X_u^{i,N})\big)^2 f(X_u^{i,N})\Big]\mathrm{d}u\leqslant C_F.
$$

Using similar computations, we also obtain $\mathbb{E}\Big[\Big|\Delta_t^{1,N}\Big|\Big]\leqslant\frac{C_F}{\sqrt{N}}$ for all $t\in[0,T]$, and thus we deduce $\lim_{N\to\infty}\mathbb{E}\big[\big|F(\bar{\mu}^N)\big|\big]=0$.

Indeed, we observe that

$$\begin{aligned}\left|\Delta_t^{1,N}\right| &\leqslant \int_0^t \int_{\mathbb{R}_+} \left|\varphi\left(X_{u-}^{1,N} + \frac{1}{N}\right) - \varphi(X_{u-}^{1,N})\right| \mathbb{1}_{\left\{z \leqslant f(X_{u-}^{1,N})\right\}} Q^1(\mathrm{d}u, \mathrm{d}z)\\ &\left|\sum_{j=1}^N \int_0^t \int_{\mathbb{R}_+} \left|\varphi\left(X_{u-}^{j,N} + \frac{1}{N}\right) - \varphi(X_{u-}^{j,N})\right| \mathbb{1}_{\left\{z \leqslant f(X_{u-}^{j,N})\right\}} \widetilde{Q}^j(\mathrm{d}u, \mathrm{d}z)\right|\\ &+ \sum_{j=1}^N \int_0^t \left(\varphi\left|X_u^{j,N} + \frac{1}{N}\right| - \varphi(X_u^{j,N}) - \frac{1}{N}\varphi'(X_u^{i,N})\right) f(X_u^{j,N})\mathrm{d}u\\ &= J_1 + J_2 + J_3.\end{aligned}$$

Since φ' is bounded, and by (5.5), there is a constant $c > 0$ such that

$$\mathbb{E}[J_i] \leqslant \frac{c}{N} \int_0^t \mathbb{E}\big[f(X_u^{1,N})\big]\mathrm{d}u \leqslant \frac{C_F}{N}.$$

Similarly, since φ'' is bounded

$$\mathbb{E}[J_3] \leqslant \frac{c}{N^2} \sum_{j=1}^N \int_0^t \mathbb{E}\big[f(X_u^{1,N})\big]\mathrm{d}u \leqslant \frac{C_F}{N}.$$

Finally using the independence between the Poisson measures $(Q_i)_{i \geqslant 1}$ and the boundness of φ', we obtain

$$\mathbb{E}\Big[(J_2)^2\Big] = \sum_{j=1}^N \int_0^t \mathbb{E}\left[\left(\varphi\left(X_u^{j,N} + \frac{1}{N}\right) - \varphi(X_u^{j,N})\right)^2 f(X_u^{j,N})\right] \leqslant \frac{C_F}{N}.$$

Combining all these bounds, we deduce that $\mathbb{E}\Big[\Big|\Delta_t^{1,N}\Big|\Big] \leqslant \frac{C_F}{\sqrt{N}}$.