4 Analysis of a diffusive interacting particle system

We present in detail in this section a diffusive model of interacting particles, studied by Benachour et al. in [BRTV98] and by Malrieu in [Mal03], which is a particular case of the laboratory example of [Szn91] presented in Section 1.1. The propagation of chaos property will be proved using a coupling method.

We consider the following nonlinear equation

$$\partial_t u = \Delta u + \operatorname{div}(u \nabla W * u), \tag{4.1}$$

where $u(t, \cdot)$ is a time-dependent probability measure on $\mathbb{R}^d$, $W : \mathbb{R}^d \to \mathbb{R}^d$ is an interaction potential, and $*$ stands for the convolution operator:

$$\nabla W * u(x) = \int \nabla W(x - y) u(\mathrm{d}y).$$

This equation arises for example in the modeling of granular media with $W(x) = |x|$ when $d = 1$. The norm of $x \in \mathbb{R}^d$ is denoted by $|x|$.
We work under the following assumptions on the potential W.

Assumption 1. *1. The potential is symmetric:* $\forall x \in \mathbb{R}^d$, $W(-x) = W(x)$.

2. *The potential is uniformly convex: there is* $\lambda > 0$ *such that* $\forall x, v \in \mathbb{R}^d$

$$\langle \mathrm{Hess} W(x) v, v \rangle \geqslant \lambda |v|^2,$$

where $\mathrm{Hess} W$ *is the hessian matrix of* W.

3. *The gradient of* W *is a locally Lipschitz-continuous function with polynomial growth: there exists a polynomial* P *such that* $\forall x, y \in \mathbb{R}^d$,

$$|\nabla W(x) - \nabla W(y)| \leqslant |x - y||P(x) + P(y)|.$$

Contrary to the assumptions of Section 1.1, the function ∇W is neither bounded nor compactly supported.

The stochastic differential equation associated with (4.1) is the following. Let $\overline{X}$ be a Markov process solving the nonlinear SDE

$$\begin{cases} \mathrm{d}\overline{X}_t = \sqrt{2}\mathrm{d}B_t - \nabla W * u_t(\overline{X}_t)\mathrm{d}t, \\ \mathrm{Law}(\overline{X}_0) = u_0, \end{cases} \tag{4.2}$$

where $u_t := \mathrm{Law}(\overline{X}_t)$ is the distribution of $\overline{X}_t$ and B is a Brownian motion. In [BRTV98, Theorem 3.1], the strong existence and uniqueness of the solution to the nonlinear SDE (4.2), when u_0 has a finite moment of order $2(1 + r)^2$, with r the degree of the polynomial growth of W, is established. Note that (4.2) can be also be written,

$$\begin{aligned} \overline{X}_t &= \overline{X}_0 + \sqrt{2}B_t - \int_0^t \nabla W * u_s(\overline{X}_s)\mathrm{d}s \\ &= \overline{X}_0 + \sqrt{2}B_t - \int_0^t \int_{\mathbb{R}^d} \nabla W(\overline{X}_s - y) u_s(\mathrm{d}y)\mathrm{d}s \end{aligned}$$

In addition, [BRTV98] proved that the moments of $\overline{X}$ are uniformly bounded in time.

Lemma 4.1 (Proposition 3.10 in [BRTV98], Lemma 5.2 in [Mal03])**.** *Let* $q \geqslant 1$*. If* $\mathbb{E}\left[|X_0|^{2q}\right]$ *is finite, then there exists* $K_q > 0$ *such that*

$$\sup_{t \geqslant 0} \mathbb{E}\left[\left|\overline{X}_t\right|^{2q}\right] \leqslant K_q.$$

■ **Exercise 3.** Using Itô's formula, show that the distribution of $\overline{X}$ is a weak solution to (4.1), in the sense that if $\mathrm{Law}(\overline{X}_t) = u_t(x)\mathrm{d}x$, we have $\forall \varphi \in \mathcal{C}_c^2(\mathbb{R}^d, \mathbb{R})$ test function of class $\mathcal{C}^2$ with compact support, $\forall t \geqslant 0$,

$$\int_{\mathbb{R}^d} \varphi(x) u_t(x) \mathrm{d}x = \int_{\mathbb{R}^d} \varphi(x) u_0(x) \mathrm{d}x + \int_0^t \int_{\mathbb{R}^d} \varphi(x)(\Delta u_s(x) + \nabla.(u_s \nabla W * u_s)(x)) \mathrm{d}s.$$

4.1 A first particle system

A natural particle system $\mathbf{X}^N = (X^{1,N}, \ldots, X^{N,N})$ associated with (4.2) is the following: for $i \in \{1, \ldots, N\}$,

$$\begin{cases} \mathrm{d}X_t^{i,N} = \sqrt{2}\mathrm{d}B_t^i - \frac{1}{N}\sum_{j=1}^N \nabla W\left(X_t^{i,N} - X_t^{j,N}\right)\mathrm{d}t, \\ X_0^{i,N} = X_0^i, \end{cases} \tag{4.3}$$

where $(B^i)_{i \geqslant 1}$ and $(X_0^i)_{i \geqslant 1}$ are two independent collections of independent Brownian motions and independent random variables with common distribution u_0 respectively. The existence of a unique strong solution to (4.3) is established in [BRTV98, Proposition 5.1]. The proof is based on an approximation of ∇W by bounded Lipschitz continuous functions and a control of the moments of the particles.

However, as we will see later, this is not the most suitable system for studying convergence to equilibrium in the long-time regime.

Asymptotic in large population Let us study the system (4.3) in the large-population limit ($N \to \infty$). We prove that propagation of chaos holds, but not uniformly in time. The proof is based on a coupling argument.

To this end, we couple the particle system (4.3) with a collection $(\overline{X}^i)$ of independent nonlinear processes driven by the same Brownian motions and starting from the same initial conditions as the interacting particles.

Let $(\overline{X}^i)_{i \geqslant 1}$ be the solution to the *nonlinear* SDE

$$\begin{cases} \mathrm{d}\overline{X}_t^i = \sqrt{2}\mathrm{d}B_t^i - \nabla W * u_t\left(\overline{X}_t^i\right)\mathrm{d}t, \\ \overline{X}_0^i = X_0^i, \end{cases}$$

where u_t is the distribution of $\overline{X}_t^i$.

Theorem 4.2 (Benachour et al. [BRTV98])**.** *If* u_0 *has a finite* 2*-order moment, then there exists a constant* $C > 0$*, such that for any* $T \geqslant 0$ *and* $N \geqslant 1$*,*

$$\sup_{0 \leqslant t \leqslant T} \mathbb{E}\left[\left|X_t^{1,N} - \overline{X}_t^1\right|^2\right] \leqslant C\frac{T^2}{N}.$$

We deduce that $(X^{1,N})_{N\geqslant 1}$ converges in distribution to $\overline{X}$. By uniqueness of the solution to the nonlinear SDE (4.2) and Proposition 2.8, the system satisfies the propagation of chaos property.

Proof of Theorem 4.2. We detail the proof when $d=1$, the case $d\geqslant 1$ is similar. We observe that

$$X_t^{i,N}-\overline{X}_t^i=-\frac{1}{N}\sum_{j=1}^N\int_0^t\int\Big(W'(X_s^{i,N}-X_s^{j,N})-W'(\overline{X}_s^i-y)\Big)u_s(\mathrm{d}y)\mathrm{d}s.$$

Then, by Itô's Formula,

$$\begin{aligned}&\sum_{i=1}^N\Big(X_t^{i,N}-\overline{X}_t^i\Big)^2\\&=-\frac{2}{N}\sum_{i,j=1}^N\int_0^t\int\Big(X_s^{i,N}-\overline{X}_s^i\Big)\Big(W'(X_s^{i,N}-X_s^{j,N})-W'(\overline{X}_s^i-y)\Big)u_s(\mathrm{d}y)\mathrm{d}s\\&=-\frac{2}{N}\sum_{i,j=1}^N\int_0^t(I_{i,j}^1+I_{i,j}^2)\mathrm{d}s,\end{aligned}$$

with

$$\begin{aligned}I_{i,j}^1&=\Big(X_s^{i,N}-\overline{X}_s^i\Big)\Big(W'(X_s^{i,N}-X_s^{j,N})-W'(\overline{X}_s^i-\overline{X}_s^j)\Big)\\I_{i,j}^2&=\Big(X_s^{i,N}-\overline{X}_s^i\Big)\int\Big(W'(\overline{X}_s^i-\overline{X}_s^j)-W'(\overline{X}_s^i-y)\Big)u_s(\mathrm{d}y)\end{aligned}$$

As W is symmetric, we have $W'(0)=0$ and $W'(x)=-W'(-x)$. Therefore, the sum of the first term gives

$$\begin{aligned}\sum_{i,j=1}^N I_{i,j}^1&=\sum_{i<j}(I_{i,j}^1+I_{j,i}^1)\\&=\sum_{i<j}\Big(X_s^{i,N}-X_s^{j,N}-\overline{X}_s^i+\overline{X}_s^j\Big)\Big(W'(X_s^{i,N}-X_s^{j,N})-W'(\overline{X}_s^i-\overline{X}_s^j)\Big).\end{aligned}$$

As the potential is uniformly convex, we deduce $(x-y)(W'(x)-W'(y))\geqslant 0$. Consequently,

$$\sum_{i,j=1}^N I_{i,j}^1\geqslant 0,$$

and we deduce that

$$\sum_{j=1}^N\Big(X_t^{i,N}-\overline{X}_t^i\Big)^2\leqslant-\frac{2}{N}\sum_{i,j=1}^N\int_0^t I_{i,j}^2\mathrm{d}s. \tag{4.4}$$

On the other hand, by the Cauchy-Schwartz inequality,

$$\begin{aligned}\left|\mathbb{E}\left[\sum_{j=1}^N I_{i,j}^2\right]\right|&=\left|\mathbb{E}\left[\Big(X_s^{i,N}-\overline{X}_s^i\Big)\sum_{j=1}^N\Big(W'(\overline{X}_s^i-\overline{X}_s^j)-W'*u_s(\overline{X}_s^i)\Big)\right]\right|\\&\leqslant\mathbb{E}\Big[\Big(X_s^{i,N}-\overline{X}_s^i\Big)^2\Big]^{1/2}h_i(s)^{1/2},\end{aligned} \tag{4.5}$$

with

$$
\begin{aligned}
h_i(s) &= \mathbb{E}\left[\left(\sum_{j=1}^{N}\Big(W'(\overline{X}_s^i - \overline{X}_s^j) - W' * u_s(\overline{X}_s^i)\Big)\right)^2\right] \\
&= \sum_{j=1}^{N}\mathbb{E}\left[\Big(W'(\overline{X}_s^i - \overline{X}_s^j) - W' * u_s(\overline{X}_s^i)\Big)^2\right],
\end{aligned}
$$

by independence of $\overline{X}^i, \overline{X}^j, \overline{X}^k$, and since $u_s = \mathrm{Law}(\overline{X}_s^j) = \mathrm{Law}(\overline{X}_s^k)$ implying that each term in the sum has a mean equal to zero. Since W' is locally Lipschitz-continuous with polynomial growth by assumption, there is $r \geqslant 0$ and $c > 0$ such that

$$
|W'(x)| \leqslant c(1 + |x|^{1+r}).
$$

By Lemma 4.1, the moments of $\overline{X}$ are uniformly bounded in time, i.e.

$$
\sup_{t\geqslant 0}\mathbb{E}[|\overline{X}_t|^{2(1+r)}] < \infty.
$$

Consequently, there is a constant $c > 0$ such that $h_i(s) \leqslant Nc^2$, and by symmetry of the system $\mathbb{E}\left[\left(X_s^{i,N} - \overline{X}_s^i\right)^2\right] = \mathbb{E}\left[\left(X_s^{1,N} - \overline{X}_s^1\right)^2\right]$. Therefore, by (4.5), we have

$$
\left|\mathbb{E}\left[\sum_{j=1}^{N} I_{i,j}^2\right]\right| \leqslant cN^{1/2}\mathbb{E}\left[\left(X_s^{1,N} - \overline{X}_s^1\right)^2\right]^{1/2}.
$$

Finally, using (4.4) and again the symmetry of the system, we obtain

$$
N\mathbb{E}\left[\left(X_t^{1,N} - \overline{X}_t^1\right)^2\right] \leqslant 2cN^{1/2}\int_0^t \mathbb{E}\left[\left(X_s^{1,N} - \overline{X}_s^1\right)^2\right]^{1/2}\mathrm{d}s
$$

and consequently, for $t \in [0, T]$,

$$
N\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\left(X_t^{1,N} - \overline{X}_t^1\right)^2\right] \leqslant 2cN^{1/2}T\left(\sup_{0\leqslant t\leqslant T}\mathbb{E}\left[\left(X_t^{1,N} - \overline{X}_t^1\right)^2\right]\right)^{1/2}
$$

and the result follows. □

Long time behavior of the particle system Since ∇W is odd, the empirical mean of the particle system (4.3), given by

$$
\frac{1}{N}\sum_{k=1}^{N} X_t^{k,N} = \frac{1}{N}\sum_{k=1}^{N} X_0^k + \frac{\sqrt{2}}{N}\sum_{k=1}^{N} B_t^k,
$$

is the sum of a random variable and a Gaussian process, with time marginal of distribution $\mathcal{N}(0, \frac{2t}{N})$. Consequently, the empirical measure does not converge when $t \to \infty$ to a probability measure. We deduce that the direction $(1, \ldots, 1)$ as a bad influence on the long time behavior of the system.

However, since the potential W is strictly convex, it known that the nonlinear process has an invariant measure, and as proved in [CMV03], the solution of(4.1) converges exponentially fast to its equilibrium.

Consequently, the above particle system does not behave as well as the nonlinear process it approximates. Following [Mal03], we introduce a new particle system, which admits an invariant measure.

4.2 A new particle system

We note that the mean of the nonlinear process $\bar{X}$ satisfies

$$\begin{aligned}\mathbb{E}[\overline{X}_t] &= \mathbb{E}[X_0] - \int_0^t \mathbb{E}\big[\nabla W * u_s(\overline{X}_s)\big]\mathrm{d}s \\ &= \mathbb{E}[X_0] - \int_0^t \mathbb{E}\Big[\nabla W(\overline{X}_s - \overline{X}'_s)\Big]\mathrm{d}s,\end{aligned}$$

where $\overline{X}'$ is an independent copy of $\overline{X}$. Since ∇W is an odd function, we deduce that $\mathbb{E}[\overline{X}_t] = \mathbb{E}[X_0]$ for any $t \geqslant 0$. To construct a particle system that provides a good approximation of the nonlinear process, it seems reasonable to introduce a system whose empirical mean is equal to the mean of u_0.

Without lost of generality, we assume in the sequel that $\int x u_0(\mathrm{d}x) = \mathbb{E}[X_0] = 0$. Recall that the system $\mathbf{X}^N$ is defined in (4.3). We introduce the process $\mathbf{Y}^N = \left(Y^{1,N}, \ldots, Y^{N,N}\right)$ on $\left(\mathbb{R}^d\right)^N$ defined by

$$Y_t^{i,N} = X_t^{i,N} - \frac{1}{N}\sum_{k=1}^N X_t^{k,N},$$

which is the projection of $\mathbf{X}^N$ on $\mathcal{H} = \left\{\mathbf{x} \in \left(\mathbb{R}^d\right)^N : \sum_{k=1}^N x^k = 0\right\}$. Then, $\forall t \geqslant 0$,

$$\frac{1}{N}\sum_{i=1}^N Y_t^{i,N} = 0.$$

We easily note that

$$X_t^{i,N} - X_t^{j,N} = Y_t^{i,N} - Y_t^{j,N},$$

and we deduce by (4.3),

$$Y_t^{i,N} = X_0^i - \frac{1}{N}\sum_{j=1}^N X_0^j + \sqrt{2}B_t^i - \frac{\sqrt{2}}{N}\sum_{j=1}^N B_t^j - \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla W(Y_s^{i,N} - Y_s^{j,N})\mathrm{d}s.$$

Uniform propagation of chaos

Theorem 4.3 (Theorem 5.1 in [Mal03]). *There exists a constant $C > 0$ such that for every $N \geqslant 1$*

$$\sup_{t \geqslant 0} \mathbb{E}\left[\left|Y_t^{1,N} - \overline{X}_t^1\right|^2\right] \leqslant \frac{C}{N}.$$

Proof. We introduce $\overline{\mathbf{Y}}^N = \left(\overline{Y}^{1,N},\dots,\overline{Y}^{N,N}\right)$ the projection of the nonlinear process $\overline{\mathbf{X}} = \left(\overline{X}^1,\dots,\overline{X}^N\right)$ on $\mathcal{H}$. We have

$$\begin{aligned}\overline{Y}^{i,N}_t &:= \overline{X}^i_t - \frac{1}{N}\sum_{i=1}^N \overline{X}^j_t \\ &= X_0^i - \frac{1}{N}\sum_{j=1}^N X_0^j + \sqrt{2}B_t^i - \int_0^t \nabla W * u_s\left(\overline{X}^i_s\right)\mathrm{d}s - \frac{\sqrt{2}}{N}\sum_{j=1}^N B_t^j + \frac{1}{N}\sum_{j=1}^N\int_0^t \nabla W * u_t(\overline{X}^j_t)\mathrm{d}s.\end{aligned}$$

Note that

$$\left\|Y_t^{i,N} - \overline{X}^i_t\right\|_{L^2} \leqslant \left\|Y_t^{i,N} - \overline{Y}^{i,N}_t\right\|_{L^2} + \left\|\overline{Y}^{i,N}_t - \overline{X}^i_t\right\|_{L^2}.$$

On one hand, by independence of the processes $(\overline{X}^i)_{i\geqslant 1}$, we have

$$\mathbb{E}\left[\left(\overline{Y}^{i,N}_t - \overline{X}^i_t\right)^2\right] = \frac{1}{N}\mathbb{E}\left[\left|\overline{X}^1_t\right|^2\right].$$

By lemma 4.1, the moments of $\overline{X}^1$ are uniformly bounded in time.
On the other hand,

$$\begin{aligned}Y_t^{i,N} - \overline{Y}^{i,N}_t = -\frac{1}{N}\int_0^t\sum_{j=1}^N\Big(\nabla W(Y_s^{i,N} - Y_s^{j,N}) - \nabla W * u_s(\overline{X}^i_s)\Big)\mathrm{d}s \\ - \frac{1}{N}\sum_{j=1}^N\int_0^t \nabla W * u_t(\overline{X}^j_t)\mathrm{d}s.\end{aligned}$$

Using Itô's formula, the same kind of decomposition as in the proof of Theorem 4.2, and that the sum of the coordinates of $\overline{Y}^{i,N}$ and $Y^{i,N}$ are equal to 0 since they belong to $\mathcal{H}$, we deduce that there is a constant $c > 0$ such that

$$\sum_{i=1}^N \mathbb{E}\left[\left|Y_t^{i,N} - \overline{Y}^{i,N}_t\right|^2\right] \leqslant -2\lambda\int_0^t\sum_{i=1}^N\mathbb{E}\left[\left|Y_s^{i,N} - \overline{Y}^{i,N}_s\right|^2\right]\mathrm{d}s + c\sqrt{N}\int_0^t\mathbb{E}\left[\left|Y_s^{i,N} - \overline{Y}^{i,N}_s\right|^2\right]^{1/2}\mathrm{d}s \tag{4.6}$$

By symmetry, we deduce

$$\mathbb{E}\left[\left|Y_t^{1,N} - \overline{Y}^{1,N}_t\right|^2\right] \leqslant -2\lambda\int_0^t\mathbb{E}\left[\left|Y_s^{1,N} - \overline{Y}^{1,N}_s\right|^2\right]\mathrm{d}s + \frac{c}{\sqrt{N}}\int_0^t\mathbb{E}\left[\left|Y_s^{1,N} - \overline{Y}^{1,N}_s\right|^2\right]^{1/2}\mathrm{d}s.$$

Introducing the function $\alpha(t) = \mathbb{E}\left[\left|Y_t^{1,N} - \overline{Y}^{1,N}_t\right|^2\right]^{1/2}$, using a similar proof as the one of the Gronwall lemma, we deduce

$$\mathbb{E}\left[\left(Y_t^{1,N} - \overline{Y}^{1,N}_t\right)^2\right]^{1/2} \leqslant \frac{c}{\lambda\sqrt{N}}(1 - \mathrm{e}^{-\lambda t}) \leqslant \frac{c}{\lambda\sqrt{N}},$$

which gives the expected upper bound. □

■ **Exercise 4.** Using the same kind of decomposition as in the proof of Theorem 4.2, prove Inequality (4.6).

Long time behavior We introduce the Wasserstein distance $\mathcal{W}_2$ between two probability measures ν and μ, with a finite second moment:

$$\mathcal{W}_2(\nu,\mu) := \left(\inf \int |x-y|^2 \pi(\mathrm{d}x,\mathrm{d}y)\right)^{1/2}$$

where the infimum is taken on the probability measures π on $\mathbb{R}^d \times \mathbb{R}^d$ with marginales ν and μ.

The potential W being strongly convex by assumption, the nonlinear system admits an invariant measure, and Carrillo, McCann and Villani [CMV03] proves that u_t convergences exponentially fast to its equilibrium u_∞ along a well-adapted functional. More precisely, let us introduce the functional

$$\eta(u) = \int u(x)\log u(x)\mathrm{d}x + \frac{1}{2}\iint W(x-y)u(x)u(y)\mathrm{d}x\mathrm{d}y.$$

Carrillo, McCann and Villani [CMV03] established that there is a constant $K>0$ such that

$$\eta(u_t) - \eta(u_\infty) \leqslant K\exp(-2\lambda t),$$

where u_∞ is the unique minimizer of η with the same mean of u_0.

Using the particle system and logarithmic Sobolev inequalities, [Mal03] obtain the speed of convergence in Wasserstein distance:

Theorem 4.4 (Theorem 1.4 in [Mal03])**.** *There is a constant $K>0$ such that*

$$\mathcal{W}_2(u_t,u_\infty) \leqslant K\mathrm{e}^{-\lambda t}.$$

Indeed, [Mal03] proves in Proposition 4.3 that the invariant measure ($t\to\infty$) of the particle system $\mathbf{Y}^N$ is given by

$$u_\infty^N = \frac{1}{z_N}\mathbb{1}_{\mathcal{H}}(y)\exp\left(\frac{1}{2N}\sum_{i,j}^N W(y_i-y_j)\right)\mathrm{d}y,$$

with $z_N = \int_{\mathcal{H}}\exp\left(\frac{1}{2N}\sum_{i,j}^N W(y_i-y_j)\right)\mathrm{d}y$, which satisfies a logarithmic Sobolev inequality with constant $\frac{2}{\lambda}$, that is for every smooth function f

$$\mathrm{Ent}_{u_\infty^N}(f^2) \leqslant \frac{2}{\lambda}\int |\nabla f|^2 \mathrm{d}u_\infty^N,$$

with $\mathrm{Ent}_\mu(f^2) := \int f^2\log f^2\mathrm{d}\mu - \int f^2\mathrm{d}\mu\,\log\left(\int f^2\mathrm{d}\mu\right)$. They deduce (see [Mal03, corollary 4.4]) the following result of the relative entropy for any $t\geqslant 0$

$$\mathrm{Ent}(u_t^N|u_\infty^N) \leqslant \mathrm{Ent}(u_0^N|u_\infty^N)\mathrm{e}^{-2\lambda t} \tag{4.7}$$

where u_t^N denotes the distribution of $\mathbf{Y}_t^N$ and the relative entropy is defined by $\mathrm{Ent}(\nu|\mu) = \int \log g\mathrm{d}\nu = \int g\log g\mathrm{d}\mu$ with g the density of ν with respect to μ: $\nu(\mathrm{d}x) = g\mu(\mathrm{d}x)$.

Ideas of the proof of Theorem 4.4. Let $N \geqslant 2$ be arbitrary. By the triangular inequality, one has

$$\mathcal{W}_2(u_t, u_\infty) \leqslant \mathcal{W}_2(u_t, u_t^{(1,N)}) + \mathcal{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) + \mathcal{W}_2(u_\infty^{(1,N)}, u_\infty),$$

where $u^{(1,N)}$ is the distribution of $Y^{1,N}$. From the uniform propagation of chaos established in Theorem 4.3, we have

$$\mathcal{W}_2(u_t, u_t^{(1,N)}) + \mathcal{W}_2(u_\infty^{(1,N)}, u_\infty) \leqslant 2\sqrt{\sup_{t\geqslant 0} \mathbb{E}\left[\left|Y_t^{1,N} - \overline{X}_t^1\right|^2\right]} \leqslant 2\frac{C}{\sqrt{N}}.$$

Note that, in the absence of uniform propagation of chaos, we cannot deduce anything about $\mathcal{W}_2(u_\infty^{(1,N)}, u_\infty)$.

We thus only have to focus on the term $\mathcal{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)})$. Note that, for μ and ν two probability measures on $\mathbb{R}^d$, we have

$$N\mathcal{W}_2(\mu, \nu)^2 \leqslant \mathcal{W}_2(\mu_N, \nu_N),$$

for any probability measures μ_N and ν_N on $\left(\mathbb{R}^d\right)^N$ with respective marginals $\mu, \dots, \mu$ and $\nu \dots, \nu$. In fact, for every $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ in $\left(\mathbb{R}^d\right)^N$, we easily observe that

$$|x - y|^2 = \sum_{i=1}^N |x_i - y_i|^2$$

and, therefore, for every measure $\pi_N = \pi^{\otimes N}$ on $\mathbb{R}^{Nd} \times \mathbb{R}^{Nd}$ with marginals μ_N and ν_N, we have

$$\int |x - y|^2 \pi_N(\mathrm{d}x, \mathrm{d}y) = N \int |x - y|^2 \pi(\mathrm{d}x, \mathrm{d}y)$$

where π is a measure on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals μ and ν.
Consequently,

$$\mathcal{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) \leqslant \frac{1}{\sqrt{N}} \mathcal{W}_2(u_t^N, u_\infty^N),$$

where u^N is the distribution of the particle system $\mathbf{Y}^N$.

We now use the following result (see also [BGL01, Corollary 3.1]).

Theorem 4.5 (Otto-Villani [OV00]). *Let μ be a absolutely continuous probability measure which satisfies a logarithmic Sobolev inequality with constant C.*
Then, for every probability measure ν absolutely continuous with respect to μ, we have

$$\mathcal{W}_2(\mu, \nu)^2 \leqslant \frac{C}{2} \mathrm{Ent}(\nu|\mu).$$

Using also (4.7), we thus deduce

$$\begin{aligned}\mathcal{W}_2(u_t^{(1,N)}, u_\infty^{(1,N)}) &\leqslant \sqrt{\frac{C}{2N} \mathrm{Ent}(u_t^N | u_\infty^N)} \\ &\leqslant \sqrt{\frac{C}{2N} \mathrm{Ent}(u_0^N | u_\infty^N)} \mathrm{e}^{-\lambda t}.\end{aligned}$$

By assumption, $u_0^N = u_0^{\otimes N}$ is the distribution of N independent random variables, and thus $\mathrm{Ent}(u_0^N|u_\infty^N)$ is of order N.
Consequently, we finally obtain the existence of a constant $C > 0$ such that

$$\mathcal{W}_2(u_t, u_\infty) \leqslant \frac{C}{\sqrt{N}} + C\mathrm{e}^{-\lambda t}.$$

We conclude by letting $N \to \infty$, since N was chosen arbitrarily. □