3 Convergence of stochastic processes

Let us recall the definition of weak convergence of probability measures.

Definition 3.1. *A sequence of probability measures $(\mu^N)_{N\geqslant 1}$ on a Polish space E converges weakly to the probability measure μ if for any $f \in \mathcal{C}_b(E)$,*

$$\lim_{N\to\infty} \int f \mathrm{d}\mu^N = \int f \mathrm{d}\mu.$$

The objective of this section is not to provide an exhaustive treatment of the convergence of continuous or càdlàg processes, but only to highlight the main results that are useful in the study of several examples of interacting particle systems. We mainly refer to the books of Billingsley [Bil99], and of Jacod and Shiryaev [JS03] for more details on the convergence of stochastic processes. We can also mention the books of Ethier and Kurtz [EK86], and of Stroock and Varadhan [SV06].

For $d \geqslant 1$ and $T > 0$, let $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ denote the space of continuous functions, and $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ denote the Skorokhod space, namely the space of *càdlàg* functions (right-continuous functions with left limits), defined on $\mathbb{R}_+$ with values in $\mathbb{R}^d$.
The Euclidean norm of a vector $x \in \mathbb{R}^d$ is denoted by $|x|$.

Endowed with suitable metrics, the spaces $E = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and $E = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ are Polish spaces (see [Bil99, JS03]). Note that the topology induced by the metric on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ that makes it a Polish space is weaker than the local uniform topology.

3.1 Tightness in $\mathcal{C}$

We start with a result ensuring the convergence for tight probability measures on $\mathcal{C}\big(\mathbb{R}_+, \mathbb{R}^d\big)$. This theorem is a consequence of Prokhorov's theorem (see Example 2.7+ Theorem 2.6 and Section 5 in [Bil99]).

Theorem 3.2. *Let $Z^N = (Z_t^N)_{t\geqslant 0}$ be a sequence of processes with values in $\mathcal{C}\big(\mathbb{R}_+, \mathbb{R}^d\big)$. Denote by μ^N the distribution of Z^N. Let μ a probability measure on $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ and Z a random process with distribution μ.*
Then the sequence $(\mu^N)_{N\geqslant 1}$ converges weakly to μ if and only if it satisfies the two following conditions

- Tightness*: the sequence $\big(\mu^N\big)$ is tight in $\mathcal{C}\big(\mathbb{R}_+, \mathbb{R}^d\big)$;*
- Convergence of the finite-dimensional distributions*: for any $k \in \mathbb{N}$ and any $(t_1, \ldots, t_k) \in \mathbb{R}_+^k$, the vector $(Z_{t_1}^N, \ldots, Z_{t_k}^N)$ converges in law to $(Z_{t_1}, \ldots, Z_{t_k})$, as $N \to \infty$.*

We now present useful results on the tightness of continuous stochastic processes. To this end, we introduce the modulus of continuity of a function $z(\cdot)$ on $[0, T]$: for $\delta \in (0, T]$

$$w_T(z; \delta) := \sup_{\substack{|s-t|\leqslant\delta \\ \text{with } s,t\in[0,T]}} |z(s) - z(t)|.$$

Note that $z \in \mathcal{C}(\mathbb{R}_+, \mathbb{R})$ if and only if $\lim_{\delta\downarrow 0} w_T(z; \delta) = 0$ for all $T \geqslant 0$ (Heine-Cantor theorem: a continuous function on a compact is uniformly continuous). We first recall a well-known result on compact sets of continuous functions.

Theorem 3.3 (Arzela-Ascoli theorem)**.** *A subset A of $\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ is relatively compact (i.e. its closure $\overline{A}$ is compact) if and only if*

(i) $\sup_{z\in A} |z(0)| < \infty$;

(ii) Equicontinuity: $\lim_{\delta\to 0} \sup_{z\in A} w_T(z;\delta) = 0$ *for all* $T \geqslant 0$.

We deduce the following characterization of the tightness of probability measures on the space of continuous functions.

Theorem 3.4 (Theorem 7.3 in [Bil99])**.** *A sequence of continuous processes $(Z^N)_{N\geqslant 1}$ on $\mathbb{R}_+$ is tight if an only if the two following conditions are satisfied:*

(i) $\forall \eta > 0$*, there exist* $a > 0$ *and* $N_0 \geqslant 1$ *such that* $\forall N \geqslant N_0$*,*

$$\mathbb{P}\big(\big|Z_0^N\big| \geqslant a\big) \leqslant \eta,$$

(ii) $\forall a, T > 0$*,*

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbb{P}\big(w_T(Z^N;\delta\big) \geqslant a) = 0.$$

Proof. Let $T > 0$. We first prove the direct implication. Assume that (Z^N) is tight. Let $\eta > 0$, and K a compact set of $\mathcal{C}([0,T], E)$ such that $\mathbb{P}(Z^N \in K) > 1 - \eta$ for any $N \geqslant 1$. We have $K \subset \{z : |z(0)| \leqslant a\}$ for a large enough, and $K \subset \{z : w_T(z;\delta) \leqslant \varepsilon\}$ for δ small enough. Thus the conditions are satisfied for $N_0 = 1$.
Assume that Conditions (i) and (ii) are satisfied. As $\mathcal{C}([0,T], E)$ is separable and complete, a finite collection of measures is tight. Let $\eta > 0$, a given by (i), then $\mathbb{P}(Z^N \in B) \geqslant 1 - \eta$ where $B := \{z : |z(0) \leqslant a|\}$. By (ii), $\mathbb{P}(Z^N \in B^k) \geqslant 1 - \eta/2^k$ for all N, with $B^k = \{z : w_T(z;\delta_k) < 1/k\}$. Let K be the closure of $A = B \cap \bigcap_k B^k$, then $\mathbb{P}(Z^N \in K) \geqslant 1 - 2\eta$. By the Arzela-Ascoli theorem, A is relatively compact, and therefore K is compact. We conclude that Z^N is tight. □

We now give a sufficient condition to ensure that condition (ii) in the previous theorem is satisfied.

Theorem 3.5 (Theorem 7.3 + Condition (7.12) in [Bil99])**.** *A sequence of continuous processes $(Z^N)_{N\geqslant 1}$ on $\mathbb{R}_+$ is tight if the two following conditions are satisfied:*

(i) $\forall \eta > 0$*, there exist* $a > 0$ *and* $N_0 \geqslant 1$ *such that* $\forall N \geqslant N_0$*,*

$$\mathbb{P}\big(\big|Z_0^N\big| \geqslant a\big) \leqslant \eta,$$

(ii) $\forall a, \eta, T > 0$ *and* $t \in [0,T]$*, there exists* $\delta \in (0,T)$ *and* $N_0 \geqslant 1$ *such that for any* $N \geqslant N_0$*,*

$$\frac{1}{\delta}\mathbb{P}\left(\sup_{t\leqslant s\leqslant t+\delta} \big|Z_t^N - Z_s^N\big| \geqslant a\right) \leqslant \eta.$$

Proof. We only have to prove that Condition (ii) in Theorem 3.4 is satisfied. Let $a, \eta, T > 0$ and $t \in [0,T]$. If $t > T - \delta$, the supremum is thus taken on $s \in [t,T]$. We set $t_k = k\delta$ for $k < \lfloor T/\delta \rfloor$. We have $(t_k - t_{k-1}) = \delta$. For a continuous function z on $[0,T]$, we note that

$$w_T(z;\delta) \leqslant 3 \max_k \sup_{s\in[t_{k-1},t_k]} |z(s) - z(t_{k-1})|,$$

and therefore,

$$\mathbb{P}(w_T(Z^N;\delta) \geqslant 3a) \leqslant \sum_{k=1}^{\lfloor T/\delta \rfloor} \mathbb{P}\left(\sup_{s\in[t_{k-1},t_k]} \left| Z^N_s - Z_{t_{k-1}} \right| \geqslant a \right)$$
$$\leqslant T\eta.$$

Taking the limit when $\eta \to 0$, we deduce that Condition (ii) in Theorem 3.4 is satisfied. □

In many examples of exchangeable particle systems, the processes $X^{1,N}$ are in L^1, thus it is sufficient to obtain good control of the first moment of $\sup_{s\leqslant t\leqslant s+\delta} \left| X^{1,N}_t - X^{1,N}_s \right|$ to obtain tightness of $(X^{1,N})_{N\geqslant 1}$.

3.2 Tightness in $\mathbb{D}$

Prove tightness for sequences of càdlàg processes is a bit more challenging because the characterization of compact sets is more tricky and because the projections on time-marginal distributions are not continuous. More precisely, the projection $p_t : z \in\mapsto z_t$ defined on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ is continuous in z if and only if $s \mapsto z_s$ is continuous in t.

Theorem 3.6 (Theorem 13.1 in [Bil99])**.** *Let $Z^N = (Z^N_t)_{t\geqslant 0}$ be a sequence of processes in $\mathbb{D}([0,T], \mathbb{R}^d)$, and Z a process in $\mathbb{D}([0,T], \mathbb{R}^d)$. We note by μ^N the distribution of Z^N.*
We say that $t \in [0,T]$ is a continuous point of Z if $t \notin \{s \in [0,T] : \mathbb{P}(Z_s - Z_{s-} \neq 0) > 0\}$.
Then, if

- *$(\mu^N)_{N\geqslant 1}$ is tight, and*
- *for any $k \in \mathbb{N}$ and any $(t_1, \dots, t_k) \in [0,T]^k$ continuous points of Z a.s., the vector $(Z^N_{t_1}, \dots, Z^N_{t_k})$ converges in distribution to $(Z_{t_1}, \dots, Z_{t_k})$ when $n \to \infty$*

Then (Z^N) converges in law to Z.

We introduce the following modulus on $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, as defined in [JS03]:

$$w'_T(z;\delta) = \inf_\pi \Big\{ \max_{1\leqslant k\leqslant r} w(z;[t_{k-1},t_k)) \text{ with } \pi = (t_0,\dots,t_r) \text{ partition of } [0,T] \text{ such that } \min_{1\leqslant k<r} (t_k - t_{k-1}) > \delta \Big\},$$

where $w(z;A) = \sup_{s,t\in E} |z(t) - z(s)|$ for $A \subset [0,T]$.

Remark 3.7. *The modulus on $\mathbb{D}([0,T], \mathbb{R})$ is defined in [Bil99] by*

$$w'_T(z;\delta) = \inf_\pi \Big\{ \max_{1\leqslant k\leqslant r} w(z;[t_{k-1},t_k)) \textit{ with } \pi = (t_0,\dots,t_r) \textit{ partition of } [0,T] \textit{ such that } \min_{1\leqslant k\leqslant r} (t_k - t_{k-1}) > \delta \Big\},$$

The main difference with the previous definition is that [JS03] doesn't require $t_r - t_{r-1} > \delta$. The reason is that, in the topology of $\mathbb{D}([0,T], \mathbb{R}^d)$, the end point $t = T$ plays a specific role, while the points $T \geqslant 0$ should not play any specific role in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$. Consequently, the proofs are slightly more involved in $\mathbb{D}([0,T], \mathbb{R}^d)$ than in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

Remark 3.8. *We have* $z \in \mathbb{D}(\mathbb{R}_+, \mathbb{R})$ *if and only if for any* $T > 0$, $\sup_{0 \leqslant t \leqslant T} |z(t)| < \infty$ *and* $\lim_{\delta \downarrow 0} w'_T(z; \delta) = 0$. *In addition, we note that*

$$w'_T(z;\delta) \leqslant w_T(z;2\delta) \quad \text{and} \quad w_T(z;\delta) \leqslant 2w'_T(z,\delta) + \sup_{t\in[0,T]} |z(t) - z(t-)|.$$

Proof. We refer to [JS03, Lemma VI.1.11] for the equivalence. We only prove here the relation between w'_T and w_T.
For the first inequality, take a partition π such that $\delta < t_k - t_{k-1} \leqslant 2\delta$.
For the second inequality, recall that $w(z;\delta) = \sup_{|t-s|\leqslant\delta} |x(t) - x(s)|$. By the condition on the partition, for $|t-s| \leqslant \delta$, there is $k \geqslant 1$ such that

- $(s,t) \in [t_{k-1}, t_k)$, and thus $|z(t) - z(s)| \leqslant w(z; [t_{k-1}, t_k))$,
- or $s < t_{k-1} \leqslant t < t_k$, and since a jump can occur in t_{k-1}, we have

$$|z(t) - z(s)| \leqslant |z(s) - z(t_{k-1})| + |x(t_{k-1}) - x((t_{k-1})-)| + |z(t) - z(t_{k-1})|$$

- or $t_{k-1} < s < t_k \leqslant t$, and since a jump can occur in t_k, we have

$$|z(t) - z(s)| \leqslant |z(s) - z(t_k)| + |x(t_k) - x(t_k-)| + |z(t) - z(t_k)|$$

We deduce that

$$|z(t) - z(s)| \leqslant 2w(z; [t_{k-1}, t_k)) + \sup_{t\in[0,T]} |z(t) - z(t-)|.$$

□

Theorem 3.9 (Ascoli's Theorem). *A subset* $A \subset \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$ *is relatively compact if and only if for any* $T \geqslant 0$

(i) $\sup_{z\in A} \sup_{t\in[0,T]} |z(t)| < \infty$*;*

(ii) $\lim_{\delta\to 0} \sup_{z\in A} w'_T(z;\delta) = 0$.

We deduce the following characterization of the tightness of distributions in $\mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

Theorem 3.10 (Theorem 13.2 in [Bil99]). *Let* $Z^N = (Z^N_t)_{t\geqslant 0}$ *be a sequence of adapted càdlàg processes on* $\mathbb{R}_+$. *The sequence* Z^N *is tight if and only if for any* $T \geqslant 0$

(i) the sequence $\left(\sup_{t\in[0,T]} \left|Z^N_t\right|\right)_{N\geqslant 1}$ *is tight:*

$$\lim_{a\to\infty} \limsup_N \mathbb{P}\left(\sup_{t\in[0,T]} \left|Z^N_t\right| \geqslant a\right) = 0;$$

(ii) $\forall a > 0$,

$$\lim_{\delta\to 0} \limsup_N \mathbb{P}\big(w'_T(Z^N, \delta) > a\big) = 0.$$

The following criterion due to Aldous, [Ald78], is one of the most classical results to prove the tightness of the laws of sequences of processes in $\mathbb{D}([0,\infty), \mathbb{R})$.

Theorem 3.11 (Aldous criterion)**.**
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$ be a filtered probability space. Let $Z^N = (Z^N_t)_{t\geqslant 0}$ be a sequence of adapted càdlàg processes on $[0,\infty)$. Assume that the sequence of processes satisfies the following conditions: for any $T \geqslant 0$

i) $\forall \eta > 0$, there exist $a > 0$ and $N_0 \geqslant 1$ such that $\forall N \geq N_0$,

$$\mathbb{P}\left(\sup_{t\leq T} |Z^N_t| \geqslant a\right) \leqslant \eta$$

(in other words, $\lim_{a\to\infty} \limsup_{N\to\infty} \mathbb{P}\left(\sup_{t\leq T} |Z^N_t| \geqslant a\right) = 0$);

ii) $\forall a > 0$,

$$\lim_{\delta\downarrow 0} \limsup_{N\to\infty} \sup_{\substack{S,S' \text{ stopping times}:\\ S\leqslant S'\leqslant S+\delta\leqslant T}} \mathbb{P}\big(|Z^N_{S'} - Z^N_S| \geqslant a\big) = 0.$$

Then the sequence of processes $\left(Z^N\right)_{N\geqslant 0}$ is tight on $\mathbb{D}([0,\infty),\mathbb{R})$.

Various proofs of the Aldous criterion can be found in the literature, see for e.g., [Bil99, Theorem 16.10], [JS03, Chapter VI, Section 4a], and [EK86, Chapter 3, Theorem 8.6].

Proof. We only need to prove that condition (ii) in Theorem 3.10 holds.
Let $T, a, \varepsilon > 0$. By assumptions, there exists $\delta(\varepsilon) > 0$ and $N(\varepsilon) \geqslant 1$ such that $\forall N \geqslant N(\varepsilon)$, for any $(\mathcal{F}^N)$- stopping times S, S' bounded by T with $S \leqslant S' \leqslant S + \delta(\varepsilon)$.

$$\mathbb{P}(\big|Z^N_{S'} - Z^N_S \geqslant a\big|) \leqslant \varepsilon. \tag{3.1}$$

We define by induction the following sequence of stopping times:

$$\begin{aligned} S^N_0 &= 0, \\ S^N_1 &= \inf\left\{t > 0 : \left|Z^N_t - Z^N_0\right| > a\right\}, \\ S^N_k &= \inf\left\{t > S^N_{k-1} : \left|Z^N_t - Z^N_{S^N_{k-1}}\right| > a\right\}. \end{aligned}$$

We notice that if $S^N_{k+1} < \infty$, then $\left|Z^N_{S^N_k} - Z^N_{S^N_{k-1}}\right| \geqslant a$. Consequently, for any $N \geqslant N_1 := N(\varepsilon)$, $k \geqslant 1$,

$$\mathbb{P}\big(S^N_k \leqslant T, S^N_k \leqslant S^N_{k-1} + \delta_1\big) \leqslant \varepsilon,$$

with $\delta_1 = \delta(\varepsilon)$.

Let $q \geqslant 1$ be such that $q\delta_1 > 2T$.

Since $S_q^N = \sum_{k=1}^q S_k^N - S_{k-1}^N$, we have for $N \geqslant N_1$,

$$\begin{aligned}
T\mathbb{P}\big(S_q^N < T\big) &\geqslant \mathbb{E}\Big[S_q^N \mathbb{1}_{\{S_q^N \leqslant T\}}\Big] = \mathbb{E}\Bigg[\sum_{k=1}^q \big(S_k^N - S_{k-1}^q\big)\mathbb{1}_{\{S_q^N \leqslant T\}}\Bigg] \\
&\geqslant \sum_{k=1}^q \mathbb{E}\Big[\big(S_k^N - S_{k-1}^N\big)\mathbb{1}_{\{S_q^N \leqslant T, S_k^N - S_{k-1}^N > \delta_1\}}\Big] \\
&\geqslant \sum_{k=1}^q \mathbb{E}\Big[\delta_1\Big(\mathbb{1}_{\{S_q^N \leqslant T\}} - \mathbb{1}_{\{S_q^N \leqslant T, S_k^N - S_{k-1}^N \leqslant \delta_1\}}\Big)\Big] \\
&\geqslant \delta_1 q\mathbb{P}\big(S_q^N \leqslant T\big) - \delta_1 q\varepsilon.
\end{aligned}$$

Since $q\delta_1 > 2T$, we deduce that for $N \geqslant N_1$,

$$\frac{1}{2}\mathbb{P}\big(S_q^N < T\big) \geqslant \mathbb{P}\big(S_q^N < T\big) - \varepsilon$$

and thus for $N \geqslant N_1$,

$$\mathbb{P}\big(S_q^N < T\big) \leqslant 2\varepsilon.$$

Using the same argument as above, we have for any $N \geqslant N_2 := \max\{N_1, N(\varepsilon/q)\}$, $k \geqslant 1$,

$$\mathbb{P}\big(S_k^N \leqslant T, S_k^N \leqslant S_{k-1}^N + \delta_2\big) \leqslant \frac{\varepsilon}{q},$$

with $\delta_2 = \delta(\varepsilon/q)$. We now consider the event

$$A^N = \big\{S_q^N \geqslant T\big\} \cap \bigcap_{k=1}^q \big\{S_k^N > \inf\big\{T, S_{k-1}^N + \delta_2\big\}\big\}.$$

We have, for $N \geqslant N_2$,

$$\begin{aligned}
\mathbb{P}((A^N)^c) &\leqslant \mathbb{P}(S_q^N < T) + \sum_{k=1}^q \mathbb{P}\big(S_k^N \leqslant T \text{ and } S_k^N - S_{k-1}^N \leqslant \delta_2\big) \\
&\leqslant 2\varepsilon + \varepsilon = 3\varepsilon
\end{aligned}$$

Let $\omega \in A^N$, and introduce $r = \inf\big\{k : S_k^N(\omega) \geqslant T\big\}$. Consider the subdivision of $[0,T]$: $0 = t_0 < t_1 < \ldots < t_r = T$ with $t_k = S_k^N(\omega)$ for $k < r$. By construction of the sequence (S_k^N), we have

$$w(Z^N(\omega); [t_{k-1}, t_k)) = \sup_{s,t \in [t_{k-1}, t_k)} \big|Z_s^N(\omega) - Z_t^N(\omega)\big| \leqslant 2a,$$

and $t_k - t_{k-1} \geqslant \delta_2$ for $k < r$ (by the definition of A^N and r). Consequently,

$$w'_T(Z^N; \delta_2) \leqslant 2a.$$

Then, the sequence is tight.

□

Definition 3.12 (Definition 3.25 in [JS03]). *A sequence $Z^N = (Z_t^N)_{t\geqslant 0}$ of processes in $\mathbb{D}\big(\mathbb{R}_+, \mathbb{R}^d\big)$ is said C-tight if it is tight and and all its limit points are supported on $\mathcal{C}\big([0,T], \mathbb{R}^d\big)$.*

Proposition 3.13 (Proposition 3.26 in [JS03]). *There is equivalence between*

(i) the sequence (Z^N) is $\mathcal{C}$-tight;

(ii) the sequence (Z^N) is tight and for all $T, a > 0$

$$\lim_{N\to\infty} \mathbb{P}\bigg(\sup_{t\leqslant T} \big|Z_t^N - Z_{t-}^N\big| > a\bigg) = 0.$$

■ **Exercise 2.** Let us recall a few properties on Poisson processes:

- $Z = (Z_t)_{t\geqslant 0}$ is a Poisson process with parameter λ if $Z_0 = 0$, Z has independent increments, and, for every $t \geqslant 0$, Z_t follows a Poisson distribution with parameter λt.
- It is also known that $M_t := Z_t - \lambda t$ and $N_t := M_t^2 - \lambda t$ are martingales with respect to the natural filtration of the Poisson process.

We consider Z^N a Poisson process with parameter $\lambda_N > 0$. By definition, $Z^N \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}_+)$. We assume that the sequence $(\lambda_N)_{N\geqslant 1}$ converges to $\lambda > 0$.

Using Aldous' criterion for the tightness, prove that $(Z^N)_{N\geqslant 1}$ converges in law to the Poisson process Z with parameter λ.

There are two main methods to prove the propagation of chaos of a symmetric interacting particle system:

- by coupling between the particle system and its asymptotic, providing a rate of convergence. It usually requires a good regularity of the coefficients, such as Lipschitz continuity.
- by tightness, consistency, and uniqueness of the nonlinear process. It can be applied to systems with nonregular coefficient, but doesn't provide any rate of convergence.

In the next section, we will detail the study of two interacting systems: the first is a diffusive interacting system and will be analyzed by coupling, and the second one is an interacting system with jumps, which will be studied by tightness method.