2 Kac's propagation of chaos

This section is mainly based on the Saint-Flour's lecture of Sznitman [Szn91], and on the recent surveys of Chaintron and Diez [CD22a, CD22b]. Please note that we refer in this document to the arXiv versions of these recent papers.

Let $\mathcal{P}(E)$ be the set of probability measures on a Polish space E (separable completely metrizable topological space). The space of bounded and continuous functions on E is denoted by $\mathcal{C}_b(E)$. For $\mu \in \mathcal{P}(E)$ and φ a test function, $\langle \mu, \varphi \rangle = \int_E \varphi \mathrm{d}\mu$.

Let $N \geqslant 2$ and consider a system $\mathbf{X}^N = \left(X^{1,N}, \dots X^{N,N}\right) \in E^N$ of N random interacting particles. We say **interacting particles** by opposition to independent variables.

In the interacting particle systems presented in Section 1, the particles are random processes defined on $\mathbb{R}_+$: $X^{i,N} = (X_t^{i,N})_{t\geqslant 0}$, which are either continuous or càdlàg. Their state spaces are therefore either $E = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$ or $E = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.

Definition 2.1. *A particle system $\mathbf{X}^N$ is said* **exchangeable** *(also called symmetric) when for any $N \geqslant 2$, the distribution of $\left(X^{i,N}\right)_{1\leqslant i \leqslant N}$ is invariant by permutation of the indexes: for any π permutation of $\{1, \cdots, N\}$, $\left(X^{1,N}, \dots, X^{N,N}\right)$ and $\left(X^{\pi(1),N}, \dots, X^{\pi(N),N}\right)$ have the same distribution.*

The exchangeability implies in particular that $X^{1,N}$ has the same distribution as any particle $X^{k,N}$. The exchangeability assumption on the system $\mathbf{X}^N$ is crucial for establishing propagation of chaos for an interacting particle system, that is, for reducing the study of the system in the large-population limit to the study of a single particle under a suitable distribution.
Provided that their initial configuration $(X_0^{i,N})_{1\leqslant i\leqslant N}$ is exchangeable, we observe that the interacting particle systems presented in Section 1 are exchangeable, since the interactions depend only on symmetric functionals of the particles, such as the empirical mean $\frac{1}{N}\sum_{i=1}^N X^{i,N}$, the empirical measure $\frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$, or the empirical cumulative distribution function $\frac{1}{N}\sum_{i=1}^N \mathbb{1}_{X^{i,N}\leqslant x}$.

Definition 2.2. *Let $\mu \in \mathcal{P}(E)$. For each $N \geqslant 2$, let μ^N denote the distribution of the system $\mathbf{X}^N$. We say that the system $\mathbf{X}^N$ is* **μ-chaotic** *if for any $k \geqslant 1$ and any $\varphi_1, \cdots, \varphi_k$ in $\mathcal{C}_b(E)$,*

$$\begin{aligned}\lim_{N\to\infty} \left\langle \mu^N, \varphi_1 \otimes \dots \varphi_k \otimes 1 \otimes \dots 1 \right\rangle &= \lim_{N\to\infty} \mathbb{E}\left[\varphi_1(X^{1,N}) \dots \varphi_k(X^{k,N})\right] \\ &= \prod_{i=1}^k \langle \mu, \varphi_i \rangle.\end{aligned}$$

We note that if $(Y_1, \dots, Y_k)$ are independent random variables with common distribution μ, then the system $\mathbf{X}^N$ is μ-chaotic if, for any $k \geqslant 1$ and any $\varphi_1, \cdots, \varphi_k$,

$$\lim_{N\to\infty} \mathbb{E}\left[\varphi_1(X^{1,N}) \dots \varphi_k(X^{k,N})\right] = \prod_{i=1}^k \mathbb{E}[\varphi_i(Y_i)] = \mathbb{E}[\varphi_1(Y_1) \dots \varphi_k(Y_k)].$$

Therefore, in the large-population limit ($N \to \infty$), we observe that k interacting particles of the system behave as k independent particles with common distribution μ.

From Definition 2.2, we easily deduce that if $\mathbf{X}^N$ is μ-chaotic, then the sequence $(X^{1,N})_{N\geqslant 1}$ converges in law to μ (take $k = 1$ in the definition). We can therefore identify the limiting distribution by studying only the asymptotic behavior of the first particle.

2.1 Criterion for propagation of chaos

We now present a useful criterion for proving that a system is chaotic.

Proposition 2.3. *Let $\mathbf{X}^N$ be an exchangeable system.*
The system $\mathbf{X}^N$ is μ-chaotic if and only if its empirical distribution $\overline{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ converges in law, as a $\mathcal{P}(E)$-valued random variable, to the deterministic probability measure μ when $N \to \infty$.

This result can be seen as a sort of Law of Large Numbers. If $\mathbf{X}^N$ is μ-chaotic, thus for any $\varphi \in \mathcal{C}_b(E)$,

$$\begin{aligned}\left\langle \overline{\mu}^N, \varphi \right\rangle - \mathbb{E}\big[\varphi(X^{1,N})\big] &= \frac{1}{N}\sum_{i=1}^N \varphi(X^{i,N}) - \mathbb{E}[\varphi(X^{1,N})] \\ &\underset{N\to\infty}{\longrightarrow} 0.\end{aligned}$$

Before proving Proposition 2.3, let us say a few words on the convergence in law of a sequence of random measures. The convergence in law is defined as the convergence against bounded continuous test functions. In the case of random measures, this representation is not intuitive.
However, we have the following useful result (proved in Section 2.2).

Proposition 2.4 (Corollary 1 in [CD22a])**.** *Let $\left(\mu^N\right)$ be a sequence of random probability measures. Consider a* <u>*deterministic*</u> *probability measure μ, i.e.* $\mathrm{Law}(\mu) = \delta_\mu$ *with $\mu \in \mathcal{P}(E)$. There is equivalence between the following assertions*

(i) (μ^N) *converge in law to μ,*

(ii) $\lim_{N\to\infty} \mathbb{E}\big[\big|\left\langle \mu^N - \mu, \varphi \right\rangle\big|\big] = 0$ *for any $\varphi \in \mathcal{C}_b(E)$.*

Note that, it is also equivalent to the convergence of $\mathbb{E}\Big[\left\langle \mu^N - \mu, \varphi \right\rangle^2\Big]$ to 0 for any $\varphi \in \mathcal{C}_b(E)$ uniformly continuous, since

$$\mathbb{E}\Big[\left\langle \mu^N - \mu, \varphi \right\rangle^2\Big] \leqslant 2\|\varphi\|_\infty \mathbb{E}\big[\big|\left\langle \mu^N - \mu, \varphi \right\rangle\big|\big] \leqslant 2\|\varphi\|_\infty \mathbb{E}\Big[\left\langle \mu^N - \mu, \varphi \right\rangle^2\Big]^{1/2}.$$

Proof of Proposition 2.3. This proposition is proved in [Szn91, Proposition 2.2] and [CD22a, Lemma 3.19].

1. Assume first that $\mathbf{X}^N$ is μ-chaotic. We prove the direct implication using Definition 2.2 with $k = 1, 2$.

   Let $\varphi \in \mathcal{C}_b(E)$. We have

$$\begin{aligned}\mathbb{E}\Big[\left\langle \overline{\mu}^N - \mu, \varphi \right\rangle^2\Big] &= \mathbb{E}\left[\left(\frac{1}{N}\sum_{i=1}^N \varphi(X^{i,N}) - \langle \mu, \varphi\rangle\right)^2\right] \\ &= \frac{1}{N^2}\sum_{i,j=1}^N \mathbb{E}\big[\varphi(X^{i,N})\varphi(X^{j,N})\big] - \frac{2}{N}\langle \mu, \varphi\rangle \sum_{i=1}^N \mathbb{E}\big[\varphi(X^{i,N})\big] + \langle \mu, \varphi\rangle^2.\end{aligned}$$

Using the exchangeability of the system, we deduce

$$\mathbb{E}\Big[\big\langle \overline{\mu}^N - \mu, \varphi\big\rangle^2\Big] = \frac{1}{N}\mathbb{E}\big[\varphi(X^{1,N})^2\big] \\ + \frac{N-1}{N}\mathbb{E}\big[\varphi(X^{1,N})\varphi(X^{2,N})\big] - 2\langle \mu, \varphi\rangle \mathbb{E}\big[\varphi(X^{1,N})\big] + \langle \mu, \varphi\rangle^2,$$

which converges to 0 when $N \to \infty$ by Definition 2.2. We deduce the convergence in law of the random measure $\overline{\mu}^N$ to the constant measure μ (by Proposition 2.4).

2. Conversely, assume that the random probability measures $\overline{\mu}^N$ converges in law to the deterministic probability measure μ.

   Let $k \geqslant 1$ and $\varphi_1, \cdots, \varphi_k$ in $\mathcal{C}_b(E)$. We have

$$\left|\big\langle \overline{\mu}^N, \varphi_1 \otimes \dots \varphi_k \otimes 1 \otimes \dots 1\big\rangle - \prod_{i=1}^k \langle \mu, \varphi_i\rangle\right| = \left|\mathbb{E}\big[\varphi_1(X^{1,N})\dots\varphi_k(X^{k,N})\big] - \prod_{i=1}^k \langle \mu, \varphi_i\rangle\right| \\ \leqslant \left|\mathbb{E}\big[\varphi_1(X^{1,N})\dots\varphi_k(X^{k,N})\big] - \mathbb{E}\left[\prod_{i=1}^k \big\langle \overline{\mu}^N, \varphi_i\big\rangle\right]\right| + \left|\mathbb{E}\left[\prod_{i=1}^k \big\langle \overline{\mu}^N, \varphi_i\big\rangle\right] - \prod_{i=1}^k \langle \mu, \varphi_i\rangle\right|. \quad (2.1)$$

   By Proposition 2.4, since $\overline{\mu}^N$ converges in law to μ, and $\varphi_i \in \mathcal{C}_b(E)$, we have the convergence of $\big\langle \overline{\mu}^N, \varphi_i\big\rangle$ to $\langle \mu, \varphi_i\rangle$ in L^1 for any $i \in \{1, \dots, k\}$. As the functions φ_i are bounded, we deduce by induction on k the convergence of the second term in (2.1) to 0 when $N \to \infty$.

   We now study the first term of (2.1).

   Using the exchangeability of the system, we note that

$$\mathbb{E}\big[\varphi_1(X^{1,N})\dots\varphi_k(X^{k,N})\big] = \frac{1}{A(k,N)} \sum_{\substack{i_1,\dots,i_k \\ \text{pairwise distinct}}} \mathbb{E}\big[\varphi_1(X^{i_1,N})\dots\varphi_k(X^{i_k,N})\big],$$

   where $A(k,N) = \frac{N!}{(N-k)!}$ is the number of pairwise distinct tuples $(i_1, \dots, i_k)$ of integers between 1 and N. In addition, by definition of $\overline{\mu}^N$,

$$\prod_{i=1}^k \big\langle \overline{\mu}^N, \varphi_i\big\rangle = \frac{1}{N^k}\prod_{i=1}^k \left(\sum_{j=1}^N \varphi_i(X^{j,N})\right) = \frac{1}{N^k}\sum_{i_1,\dots,i_k} \varphi_1(X^{i_1,N})\dots\varphi_k(X^{i_k,N}) \\ = \frac{1}{N^k} \sum_{\substack{i_1,\dots,i_k \\ \text{pairwise distinct}}} \varphi_1(X^{i_1,N})\dots\varphi_k(X^{i_k,N}) + R(k,N),$$

   Denoting by $C = \sup_{1\leqslant i\leqslant k} \|\varphi_i\|_\infty$, we note that the remainder term $R(k,N)$ satisfies

$$|R(k,N)| \leqslant \frac{C^k}{N^k}\big(N^k - A(k,N)\big).$$

Consequently,

$$\begin{aligned}
&\left|\mathbb{E}\big[\varphi_1(X^{1,N})\dots\varphi_k(X^{k,N})\big] - \mathbb{E}\left[\prod_{i=1}^{k}\langle\overline{\mu}^N,\varphi_i\rangle\right]\right| \\
&\leqslant \left(\frac{1}{A(k,N)}-\frac{1}{N^k}\right)A(k,N)C^k + \mathbb{E}[|R(k,N)|] \\
&\leqslant 2C^k\left(1-\frac{A(k,N)}{N^k}\right) \leqslant 2C^k\left(1-\left(1-\frac{k-1}{N}\right)^k\right) \leqslant 2C^k\frac{k(k-1)}{N},
\end{aligned}$$

which goes to 0 when $N\to\infty$. Then $\mathbf{X}^N$ is μ-chaotic.

□

2.2 About the convergence in law of random measures

For completeness of this section, we now prove Proposition 2.4. Recall that weak convergence of measures in $\mathcal{P}(\mathcal{P}(E))$ is defined as the convergence against test functions in $\mathcal{C}_b(\mathcal{P}(E))$. An example of such functionals is given by the linear functions: $\mu\mapsto\langle\mu,\varphi\rangle$ for a given $\varphi\in\mathcal{C}_b(E)$. However, the weak convergence of measures is not simple to represent.

Note that for a random variable X on E, we usually denote by $\mu\in\mathcal{P}(E)$ its distribution: $\mu = \mathrm{Law}(X)$. For a $\mathcal{P}(E)$-valued random variable μ, we denote here by $\widehat{\mu}\in\mathcal{P}(\mathcal{P}(E))$ its distribution: $\widehat{\mu} = \mathrm{Law}(\mu)$.

We introduce various distances on $\mathcal{P}(E)$ and $\mathcal{P}(\mathcal{P}(E))$. First, the distance $\mathcal{D}$ on $\mathcal{P}(E)$, defined by

$$\mathcal{D}(\mu,\nu) = \sum_{k=1}^{\infty}\frac{1}{2^k}|\langle\mu-\nu,\varphi_k\rangle|,$$

where $(\varphi_k)_{k\in\mathbb{N}}$ is a family of continuous bounded functions on E with $\|\varphi_k\|_\infty\leqslant 1$, and satisfying the *separating property* (see [EK86, Section III-4]): $\forall\mu,\nu\in\mathcal{P}(E)$,

$$\Big(\forall k\in\mathbb{N}\quad \langle\mu,\varphi_k\rangle = \langle\nu,\varphi_k\rangle\Big) \quad\Longrightarrow\quad \Big(\mu=\nu\Big),$$

and such that $\mathcal{D}$ metrizes the topology of weak convergence on $\mathcal{P}(E)$. Since E is a Polish space such a family exists. When E is a compact Polish space, it is a consequence of the Stone-Weierstrass theorem, and when E is a general Polish space, we refer to [SV06, Theorem 1.1.2]).

Set $\mathcal{E} := \mathcal{P}(E)$. For $\widehat{\mu}\in\mathcal{P}(\mathcal{E})$ and $\widehat{\nu}\in\mathcal{P}(\mathcal{E})$, let $\mathcal{W}_{\mathcal{D}}$ the Wasserstein distance on $\mathcal{P}(\mathcal{E})$ defined by

$$\mathcal{W}_{\mathcal{D}}(\widehat{\mu},\widehat{\nu}) := \inf_{\mu\sim\widehat{\mu},\nu\sim\widehat{\nu}}\mathbb{E}[\mathcal{D}(\mu,\nu)] = \inf_{\widehat{\pi}\in\Pi(\widehat{\mu},\widehat{\nu})}\int_{\mathcal{E}\times\mathcal{E}}\mathcal{D}(\mu,\nu)\widehat{\pi}(\mathrm{d}\mu,\mathrm{d}\nu)$$

with $\Pi(\widehat{\mu},\widehat{\nu})$ is the set of probability measures $\widehat{\pi}$ on $\mathcal{E}\times\mathcal{E}$ with marginals $\widehat{\mu}$ and $\widehat{\nu}$.

We first prove the following result.

Proposition 2.5 (Proposition 6 in [CD22a]). *Let $\left(\mu^N\right)_{N\in\mathbb{N}}$ be a sequence of random probability measures and μ be a random probability measure.*

(i) if $\lim_{N\to\infty}\mathcal{W}_{\mathcal{D}}\big(\mathrm{Law}(\mu^N),\mathrm{Law}(\mu)\big)=0$, then the sequence $(\mu^N)_{N\geqslant 0}$ of $\mathcal{P}(E)$-valued random variables converges in law towards μ.

(ii) if $\forall\varphi\in\mathcal{C}_b(E)$, $\lim_{N\to\infty}\mathbb{E}\big[\big|\big\langle\mu^N-\mu,\varphi\big\rangle\big|\big]=0$, then

$$\lim_{N\to\infty}\mathcal{W}_{\mathcal{D}}(\mathrm{Law}(\mu^N),\mathrm{Law}(\mu))=0.$$

(iii) If for any $\varphi\in\mathcal{C}_b(E)$, we have

$$\lim_{N\to\infty}\mathbb{E}\big[\big|\big\langle\mu^N-\mu,\varphi\big\rangle\big|\big]=0,$$

then $\left(\mu^N\right)_{N\in\mathbb{N}}$ converges in law to μ.

Proof. (i) Recall that the convergence in law is characterized by the convergence of integrals against test functions in the set of bounded uniformly continuous functions (Portmanteau theorem, see [Bil99, Theorem 2.1]). Let $\Phi\in\mathcal{C}_b(\mathcal{E})$ be a function uniformly continuous for the metric $\mathcal{D}$: for any $\varepsilon>0$, there exists $\delta:=\delta(\varepsilon)>0$ such that for any $\mu,\nu\in\mathcal{E}$,

$$\mathcal{D}(\mu,\nu)\leqslant\delta\quad\Rightarrow\quad|\Phi(\mu)-\Phi(\nu)|\leqslant\varepsilon.$$

Recall the well-known relation for a random variable X: $\langle Law(X),\Phi\rangle=\int\Phi\mathrm{d}\mathbb{P}_X=\mathbb{E}[\Phi(X)]$. Consequently, we have

$$\begin{aligned}\big|\big\langle\mathrm{Law}(\mu^N)-\mathrm{Law}(\mu),\Phi\big\rangle\big|&\leqslant\mathbb{E}\big[\big|\Phi(\mu^N)-\Phi(\mu)\big|\big]\\&\leqslant\varepsilon+2\|\Phi\|_\infty\mathbb{P}(\big|\Phi(\mu^N)-\Phi(\mu)\big|\geqslant\varepsilon)\\&\leqslant\varepsilon+2\|\Phi\|_\infty\mathbb{P}(\mathcal{D}\big(\mu^N,\mu\big)>\delta)\\&\leqslant\varepsilon+\frac{2\|\Phi\|_\infty}{\delta}\mathbb{E}\big[\mathcal{D}\big(\mu^N,\mu\big)\big],\end{aligned}\tag{2.2}$$

where the uniform continuity of Φ is used in the third line, and Markov's inequality in the last line. Taking the infimum on every pair of random measures (μ^N,μ) with distribution in $\Pi\big(\mathrm{Law}(\mu^N),\mathrm{Law}(\mu)\big)$, we deduce

$$\big|\big\langle\mathrm{Law}(\mu^N)-\mathrm{Law}(\mu),\Phi\big\rangle\big|\leqslant\varepsilon+\frac{2\|\Phi\|_\infty}{\delta}\mathcal{W}_{\mathcal{D}}(\mathrm{Law}(\mu^N),\mathrm{Law}(\mu)).$$

The result of (i) follows by taking first the limit when $N\to\infty$, and then when $\varepsilon\to 0$. We thus recover the fact that the topology induced by Wasserstein distance is stronger than the one induced by the weak convergence.

(ii) By definition of $\mathcal{W}_{\mathcal{D}}$ and the monotone convergence theorem, we have

$$\begin{aligned}\mathcal{W}_{\mathcal{D}}(\mathrm{Law}(\mu^N),\mathrm{Law}(\mu))&\leqslant\mathbb{E}[\mathcal{D}(\mu^N,\mu)]\\&\leqslant\sum_{k=1}^{\infty}\frac{1}{2^k}\mathbb{E}[\big|\big\langle\mu^N-\mu,\varphi_k\big\rangle\big|].\end{aligned}$$

Using the dominated convergence theorem, the conclusion of (ii) follows.

(iii) if $\forall \varphi \in \mathcal{C}_b(E)$ uniformly continuous, $\lim_{N\to\infty} \mathbb{E}\big[\big|\big\langle \mu^N - \mu, \varphi \big\rangle\big|\big] = 0$, (ii) implies $\lim_{N\to\infty} \mathcal{W}_{\mathcal{D}}(\mathrm{Law}(\mu^N), \mathrm{Law}(\mu)) = 0$, and thus, by (i), $(\mu^N)_{N\geqslant 0}$ converges in law to μ.

The proposition is proved. □

We are now able to prove Proposition 2.4.

Proof of Proposition 2.4. The implication $(ii) \Rightarrow (i)$ is given by Proposition 2.5.
To prove the direct implication $(i) \Rightarrow (ii)$, we observe that for a fixed $\varphi \in \mathcal{C}_b(E)$, when μ is deterministic the function $\Phi : \nu \mapsto |\langle \nu - \mu, \varphi\rangle|$ is a deterministic function on $\mathcal{P}(\mathcal{E})$. In addition, Φ is bounded by $2\|\varphi\|_\infty$ and continuous for the weak topology on $\mathcal{P}(E)$:

$$|\Phi(\nu_1) - \Phi(\nu_2)| = ||\langle \nu_1 - \mu, \varphi\rangle| - |\langle \nu_2 - \mu, \varphi\rangle|| \leqslant |\langle \nu_1 - \nu_2, \varphi\rangle|,$$

by the reverse triangle inequality. Consequently, it is a test function for the convergence in law, and since (μ^N) converges in law to μ, the result follows. □

2.3 Tightness

We first recall the definition of tightness.

Definition 2.6. *A sequence of probability measures $(\mu^N)_{N\geqslant 1}$ on E is* **tight** *if $\forall \varepsilon > 0$ there is a compact set K_ε of E such that $\sup\limits_{N\geqslant 1} \mu^N(K_\varepsilon^c) \leqslant \varepsilon$, where K_ε^c is the complement of K_ε.*

By abuse of notation, we say that a sequence of random varables $(Z^N)_{N\geqslant 1}$ is tight if the sequence of their distributions $(\mathrm{Law}(Z^N))_{N\geqslant 1}$ is tight.

The notion of tightness for a sequence of probability measures is related to the notion of relative compactness for the topology induced by weak convergence. The analogous notion for a numerical sequence $(u_N)_{N\geqslant 1}$ is boundedness. It is well known that every bounded sequence admits a convergent subsequence. Moreover, if the limit is unique, then the *whole* sequence $(u_N)_{N\geqslant 1}$ converges. Similar properties hold for tight sequences of probability measures, as we will see later.
When E is a polish space, we have the following result proved in [Bil99, Section 5] (only E metric space is needed for the direct implication).

Theorem 2.7 (Prokhorov theorem)**.** *A sequence of probability measures $\left(\mu^N\right)_{N\geqslant 1}$ on E is tight if and only if the sequence $\left(\mu^N\right)_{N\geqslant 1}$ is relatively compact.*

In the case of exchangeable interacting particle systems, we have the following result, which allows us to work with the distribution of a single particle rather than with the empirical distribution of the whole particle system.

Proposition 2.8. *Let $\mathbf{X}^N$ be an exchangeable system with values in E^N. The sequence of random measures $\overline{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ is tight if and only if the sequence $\left(\mathrm{Law}(X^{1,N})\right)_{N\geqslant 1}$ is tight.*

Let us introduced the **intensity measure** I_μ of a random measure μ as the probability measure in $\mathcal{P}(E)$ defined by

$$\langle I_\mu, \varphi\rangle = \mathbb{E}[\langle \mu, \varphi\rangle] = \int_{\mathcal{P}(E)} \langle \nu, \varphi\rangle \widehat{\mu}(\mathrm{d}\nu),$$

for $\varphi \in \mathcal{C}_b(E)$, where $\widehat{\mu} = \mathrm{Law}(\mu)$. We have the following result linking the tightness of random measures and the tightness of their intensity measures.

Lemma 2.9 (Lemma 3.15 in [CD22a])**.** *The tightness of a sequence of random measures* $(\mu^N)_{N\geqslant 1}$*, as* $\mathcal{P}(E)$*-valued random variables, is equivalent to the tightness of the sequence of their intensity measures* $(I^N)_{N\geqslant 1}$*.*

We easily see that Proposition 2.8 is a direct consequence of the above lemma, because the intensity measure I^N of $\overline{\mu}^N = \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$, by exchangeability of the system $\mathbf{X}^N$, is

$$\left\langle I^N, \varphi\right\rangle = \mathbb{E}[\varphi(X^{1,N})].$$

In this case, the intensity measure is just the law of $X^{1,N}$.
Note that the weak convergence of a sequence of distributions of random measures $(\mu^N)_{N\geqslant 1}$ in $\mathcal{P}(\mathcal{P}(E))$ is not equivalent to the weak convergence of their intensity measures $(I^N)_{N\geqslant 1}$ in $\mathcal{P}(E)$ (only the direct implication holds, see Lemma 3.14 in [CD22a]). Note that, in the case of a chaotic exchangeable system $\mathbf{X}^N$, the asymptotic measure μ is also the limiting distribution of the first particle $(X^{1,N})_{N\geqslant 1}$. When $E = \mathcal{C}(\mathbb{R}_+, \mathbb{R}^d)$, using [Bil99, Theorem 5.1] and its corollary, the weak convergence of $(\mathrm{Law}(X^{1,N}))_{N\geqslant 0}$ to a measure μ in $\mathcal{P}(\mathcal{C}(\mathbb{R}_+, \mathbb{R}^d))$ is equivalent to $(\mathbf{X}^N)_{N\geqslant 1}$ μ-chaotic (see also [Bil99, Examples 5.1 and 5.2]). The situation is more complicated when working in the space $E = \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$, where the projection on the time marginals are not continuous, and therefore the limiting distributions have to be characterized with the limiting behavior of the empirical measure $\overline{\mu}^N$.

Proof of Lemma 3.5. We refer to the proofs of [Szn91, Proposition 2.2] and [CD22a, Lemma 3.15]. We note that the function $\mu \mapsto \langle \mu, \varphi\rangle$ is a continuous function, bounded by $\|\varphi\|_\infty$, for any $\varphi \in \mathcal{C}_b(E)$. Therefore, we easily deduce the continuity of $\mu \mapsto I_\mu$, and the direct implication follows. Consequently, we only need to prove that the tightness of I^N implies the tightness of μ^N.
Let $\varepsilon > 0$, and K_ε be a compact subset of E such that $\sup_{N\geqslant 1} I^N(K_\varepsilon^c) \leqslant \varepsilon$.
Let $\eta > 0$. We note that by the Markov inequality, for any $N \geqslant 1$,

$$\begin{aligned}\widehat{\mu}^N\left(\left\{\nu \in \mathcal{P}(E) : \nu(K_{\varepsilon\eta}^c) \geqslant \eta\right\}\right) = \int \mathbb{1}_{\left\{\nu(K_{\varepsilon\eta}^c)\geqslant\eta\right\}}\widehat{\mu}^N(\mathrm{d}\nu) \leqslant \frac{1}{\eta}\int \nu\left(K_{\varepsilon\eta}^c\right)\widehat{\mu}^N(\mathrm{d}\nu)\\ = \frac{1}{\eta}\int \left\langle \nu, \mathbb{1}_{K_{\varepsilon\eta}^c}\right\rangle \widehat{\mu}^N(\mathrm{d}\nu) = \frac{1}{\eta} I^N\left(K_{\varepsilon\eta}^c\right) \leqslant \varepsilon.\end{aligned}$$

We deduce that

$$\widehat{\mu}^N\left(\bigcup_{p\geqslant 1}\left\{\nu \in \mathcal{P}(E) : \nu(K^c_{(\varepsilon 2^{-p})/p}) > 1/p\right\}\right) \leqslant \varepsilon \sum_{p\geqslant 1} 2^{-p} = \varepsilon$$

Let $\mathcal{K}_\varepsilon = \bigcap_{p\geqslant 1}\left\{\nu \in \mathcal{P}(E) : \nu(K^c_{\varepsilon 2^{-p}/p}) \leqslant 1/p\right\}$. $\mathcal{K}_\varepsilon$ is a compact subset of $\mathcal{P}(E)$ and we deduce that $\left(\widehat{\mu}^N\right)_{N\geqslant 1}$ is tight. □

From Proposition 2.8, and as we will see in the next section, we deduce that in order to prove that a propagation of chaos holds for an exchangeable system $\mathbf{X}^N$, it is sufficient to

- prove the tightness of the sequence of first particle's distributions $(X^{1,N})_{N\geqslant 1}$,
- identify the limiting distribution, and

- establish the uniqueness of this limit.

In Section 1, several interacting particle systems were introduced, where $\mathbf{X}^N = (X_t^{1,N}, \dots, X_t^{N,N})_{t \geqslant 0}$ are càdlàg processes with values in $\mathbb{R}^d$, $d \geqslant 1$. In the next section, we recall some properties of weak convergence and tightness for such processes. In Sections 4 and 5, we analyze two examples in details and, whenever possible, provide an estimate of the rate of convergence to the asymptotic distribution. The long-time behavior of the system and of its asymptotic distribution are also studied.