# Stochastic models for rank-based interactions

Raphaël Tran-Thanh

advised by Hélène Guérin[1] and Dante Mata López[1]

March - July 2025

[1]UQÀM – Université du Québec à Montréal

# Contents

# 1 Introduction

## 1.1 Rank-based interactions in a portfolio

In the early 2000s, E. R. Fernholz introduced a class of stochastic models for equity markets in which stock portfolios are subject to rank-based interactions. By modeling the evolution of a stock as a diffusion process, we assume that each stock's drift and volatility coefficients depend on its rank by market capitalization. The simplest such model is known as the standard Atlas model and was introduced in [Fer02].

Figure 1: Farnese Atlas (2nd century AD)

In this setting, all stocks are of zero drift except for the smallest which drives the market. Because the growth of the whole portfolio is supported only by this smallest stock, the model is named after the Titan Atlas, eternally holding up the sky. Such rank-based stochastic models successfully capture an empirical feature of real equity markets, which is the so-called small-cap premium. Since they represent riskier investments and have more potential, stocks with smaller capitalization tend to exhibit higher growth rates, a phenomenon which conventional models often fail to capture. Formally, the rank dependency in Atlas models results in nonlinear, piecewise constant coefficients for the set of stochastic differential equations (SDEs) which rule the behavior of the portfolio. As a result, existence and uniqueness of solutions are fairly difficult to prove. Notable results by Fernholz *et al.* [FIKP12], and by Ichiba, Karatzas and Shkolnikov [IKS13] provide answers to these questions. Jourdain and Reygner [JR13] further investigated conditions allowing propagation of chaos in a rank-based particle system. Building on these advances, we now turn our attention to extending the rank-based framework to non-financial settings.

## 1.2 Motivation and objectives

We intend to transpose the framework of rank-based interactions from modeling the growth of equity stocks to such behaviors as dominant and dominated within a population, *e.g.* in modeling the individual weight of fish in a shoal. The first objective of this report is bibliographical. It aims to give an understanding of the usual methods used to show results of existence and uniqueness of solutions for complex stochastic differential systems (SDS).

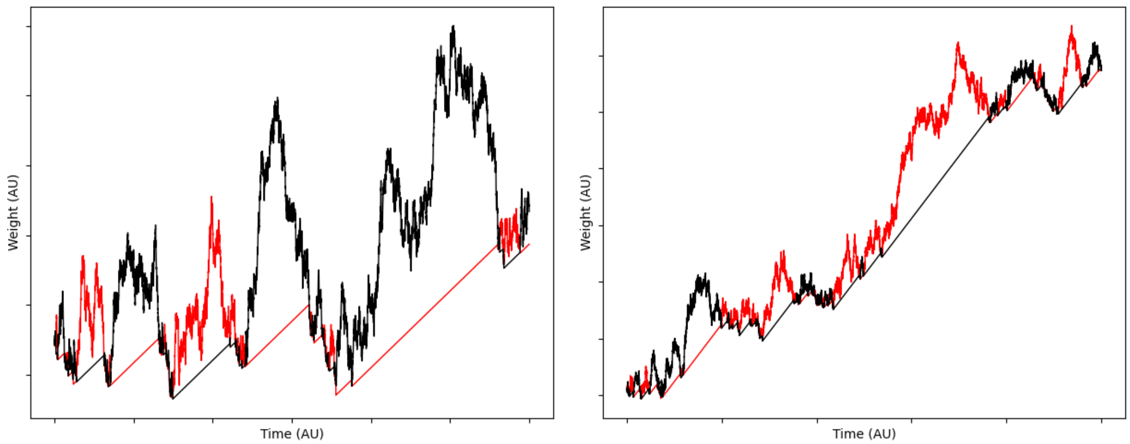


Figure 2: Toy simulation of the Atlas model (100 and 1000 iterations).

Diving into the details of the methods used in [FIKP12] and [IKS13] will give us an overview on the results we can expect when working with such rank-based processes. Working with weak or strong solutions, however, calls for significantly different approaches and mathematical tools. In particular, the search for weak solutions as is presented by Bass and Pardoux [BP87] sheds another light on how to tackle these issues. Looking into [JR13] was then our ground motivation for investigating chaos propagation properties in a more general setting of rank-based SDEs. In existing works, the drift and volatility coefficients are assumed to be piecewise constant. In real-life populations, however, individual characteristics conceivably vary over seasonal fluctuations and life phases. We are thus looking to extend this setting by including both time and space dependencies. The main focus of my contribution to this research topic is to prove that, for piecewise smooth coefficients, we still have propagation of chaos in these particle systems, *i.e.* a form of asymptotic independence and uniformity among the individuals of the population as its size increases. By formulating the mean-field study of our stochastic system as a problem of partial differential equation (PDE) analysis, we will translate the initial question to showing the existence of a unique weak solution to a certain Cauchy problem. As we shall see in Section 3.3, we were unable to prove such uniqueness in the generalized setting. However, the leads presented in this report show promising signs that propagation of chaos can be achieved in this context. Hélène, Dante and I intend to pursue our investigation in the hopes of reaching a successful conclusion.

# 2 The Atlas model - Weak and strong solutions

## 2.1 Preliminaries on SDEs

### 2.1.1 Weak and strong solutions to a diffusion equation

In this work, we will put an emphasis on the distinction between weak and strong solutions to an SDE. Let us thus begin by giving a proper definition of existence and uniqueness of solutions in both cases. Consider the following diffusion process:

$$\mathrm{d}X_t = b(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t, \tag{2.1}$$

denoted $E(b, \sigma)$, where $B$ is a standard Brownian motion, and $b$, $\sigma$ are measurable functions. The following definitions can be found in [KS91].

**Definition 2.1.1** (Strong solution)**.** *Fix a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. A strong solution to $E(b, \sigma)$ w.r.t. a fixed Brownian motion $B$ and initial condition $x_0$ is an $\mathcal{F}_t$-adapted, continuous process $X$ such that:*

*(i)* $\mathbb{P}[X_0 = x_0] = 1$;

*(ii)* $\forall t \geq 0$, $\mathbb{P}\Big[\int_0^t (|b(s, X_s)| + \sigma^2(s, X_s))\,\mathrm{d}s < \infty\Big] = 1$;

*(iii) The integral version of (2.1) holds $\mathbb{P}$-almost surely:*

$$\forall t \geq 0, \;\; X_t = x_0 + \int_0^t b(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dB_s.$$

Let us now give an appropriate definition for uniqueness in a strong sense:

**Definition 2.1.2** (Pathwise uniqueness)**.** *We say that pathwise uniqueness holds for equation $E(b, \sigma)$ if, and only if, given a standard Brownian motion $B$, and an initial condition $x_0$: if $X$ and $\tilde{X}$ are two strong solutions of $E(b, \sigma)$, then,*

$$\mathbb{P}\Big[X_t = \tilde{X}_t;\, 0 \leq t < \infty\Big] = 1.$$

The above conditions in 2.1.1 define a strong solution $X$ as a measurable functional of an input Brownian motion $B$ as well as an initial condition $x_0$. On the other hand, a weak solution to the diffusion equation $E(b, \sigma)$ is rather defined relative to a specific Brownian motion and, in some sense, more "distributional".

**Definition 2.1.3** (Weak solution)**.** *A weak solution to $E(b, \sigma)$ is defined as a triple composed of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ of sub-$\sigma$-fields of $\mathcal{F}$, and a pair of processes $(X, B)$, such that:*

*(i)* $X = (X_t)_{0 \leq t < \infty}$ *is a continuous, $\mathcal{F}_t$-adapted process,*
$B = (B_t)_{0 \leq t < \infty}$ *is a standard Brownian motion;*

*(ii) Conditions (ii) and (iii) of Definition 2.1.1 are satisfied.*

**Definition 2.1.4** (Uniqueness in distribution)**.** *We say that uniqueness in distribution holds for equation $E(b, \sigma)$ if, for any two weak solutions $(\Omega, \mathcal{F}, \mathbb{P})$, $(\mathcal{F}_t)_{0 \le t < \infty}$, $(X, B)$, and $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, $(\tilde{\mathcal{F}}_t)_{0 \le t < \infty}$, $(\tilde{X}, \tilde{B})$ with the same initial distribution, i.e.,*

$$\mathcal{L}(X_0) = \mathcal{L}(\tilde{X}_0),$$

*then $X$ and $\tilde{X}$ have the same distribution.*

Keeping in mind that a weak solution is given by a triple, we will often abuse the terminology by calling a given weak solution $X$. Since weak solutions to an SDE might be defined on different probability spaces, there is no immediate way to compute probabilities of the form:

$$\mathbb{P}\Big[X_t = \tilde{X}_t;\ 0 \le t < \infty\Big],$$

for two different weak solutions $X$ and $\tilde{X}$. In this regard, there is no sense in trying to show pathwise uniqueness in the case of weak solutions in general. The notion of uniqueness in distribution described in Definition 2.1.4 is thereby much better adapted in the weak context. The following simple example gives an idea for the nuance between weak and strong solutions, and their associated sense of uniqueness. Consider the following trivial SDE:

$$\mathrm{d}X_t = \mathrm{d}B_t. \tag{2.2}$$

Let $W$ and $\tilde{W}$ be two standard Brownian motions, not necessarily defined on the same probability space, and let $X := W$ and $\tilde{X} := \tilde{W}$. Then, with the initial condition $X_0 = 0$, $X$ and $\tilde{X}$ define two weak solutions to (2.2). Indeed, since the driving Brownian motion is not given, we can set $B := W$ and $\tilde{B} := \tilde{W}$, and write:

$$\mathrm{d}X_t = dB_t,\ \ \mathrm{d}\tilde{X}_t = \mathrm{d}\tilde{B}_t,$$

yet there is no sense in computing $\mathbb{P}\Big[X_t = \tilde{X}_t;\ 0 \le t < \infty\Big]$ when $W$ and $\tilde{W}$ are not defined on the same probability space. Furthermore, even if the weak solutions are defined on the same probability space, say $X := W$ and $\tilde{X} := -W$, we still cannot expect pathwise uniqueness to hold. Let $B := W$ and $\tilde{B} := -W$, then with the same initial condition, $X$ and $\tilde{X}$ still define two weak solutions to (2.2), and yet, for all $t \ge 0$:

$$\mathbb{P}\Big[X_t = \tilde{X}_t;\ 0 \le t < \infty\Big] = 0.$$

As one may imagine, finding strong, pathwise unique solutions to complex SDEs is usually more difficult than proving weak existence and uniqueness. Thankfully, we will see in the following that weak solutions are powerful, and sufficient in the context of chaos propagation. Working with weak solutions also allows for weaker assumptions on the drift coefficient $b$.

Let us now state some of the key results which we will need in order to understand the construction of both strong and weak solutions to the equations of the standard Atlas model.

### 2.1.2 Tanaka's formula and cumulative local times

In [FIKP12], the authors exhibit a strong solution to the Atlas equations, the construction of which relies on the introduction of the local time at zero of a certain process. This local time can be defined as follows, through Tanaka's formula, taken at the origin.

**Theorem 2.1.5** (Tanaka's formula)**.** *For any continuous semimartingale $Y$, there exists a càdlàg process (up to a modification) $(L_t^Y)_t$, which is increasing, such that for all $t \geq 0$:*

$$|Y_t| = |y| + \int_0^t sgn(Y_s)ds \ + L_t^Y. \tag{2.3}$$

*With the convention $sgn(Y_s) = \mathbf{1}_{Y_s>0} - \mathbf{1}_{Y_s\leq 0}$, which is then the left derivative of the function $|.|$.*

Intuitively, $L_t^Y$ arises from an extension of Itô's formula to functions that are not $C^2$, but only left differentiable (or, *e.g.*, convex functions). It is called the cumulative local time at the origin over $[0,t]$ of the process $Y$ and can be interpreted as the time spent "at 0" by the process.

One can also find in the literature (*e.g.* [KS84]) the following equivalent definition:

$$L_t^Y = \lim_{\epsilon \to 0} \frac{1}{4\epsilon} \int_0^t \mathbf{1}_{-\epsilon < Y_s < \epsilon} ds. \tag{2.4}$$

which makes the intuitive interpretation of the local time at 0 clearer.

### 2.1.3 Change of measure and the Cameron-Martin-Girsanov (CMG) Theorem

In this section we present a key result in understanding how stochastic processes behave under a change of reference probability. There are many versions of the Girsanov, or CMG theorem, the first of which appeared in [CM44]. Here, we present such a result in the case of a real-valued, one-dimensional Brownian motion.

Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a one-dimensional standard Brownian motion $B = \{B_t;\ 0 \leq t < \infty\}$ (with respect to $\mathbb{P}$) with its associated filtration $\{\mathcal{F}_t^B\}_{t\geq 0}$. For all $t \geq 0$, $\mathcal{F}_t^B$ is a subalgebra of $\mathcal{F}$. Next, let $X = \{X_t;\ 0 \leq t < \infty\}$ be a real-valued, $\mathcal{F}_t$-adapted process. Assume that for some constant $T > 0$ we have:

$$\int_0^T X_t^2 \, dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

Having $X_t$, let us introduce:

$$M_t := e^{Y_t}, \text{ where } Y_t := -\frac{1}{2}\int_0^t X_s^2 \, ds + \int_0^t X_s \, dB_s. \tag{2.5}$$

Note that $M_t > 0$, $\mathbb{P}$-almost surely. In differential notation, we have:

$$dY_t = -\frac{1}{2}X_t^2\,dt + X_t\,dB_t, \quad Y_0 = 0. \tag{2.6}$$

Applying Itô's formula to (2.5) yields:

$$dM_t = e^{Y_t}dY_t + \frac{1}{2}e^{Y_t}X_t^2\,dt = M_tX_t\,dB_t, \quad M_0 = 1. \tag{2.7}$$

Of course, (2.7) can be written equivalently as an integral equation:

$$M_t = 1 + \int_0^t M_sX_s\,dB_s. \tag{2.8}$$

It follows that $M_t$ is an $\mathcal{F}_t$-martingale (with respect to $\mathbb{P}$). In particular, for all $t \in [0, T]$,

$$\mathbb{E}^{\mathbb{P}}[M_t] = \mathbb{E}^{\mathbb{P}}[M_0] = 1. \tag{2.9}$$

Having $M_t$, and in view of (2.9), we introduce the following probability measures on $(\Omega, \mathcal{F})$:

$$\mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[M_T\mathbf{1}_A], \quad \mathbb{Q}_t(A) = \mathbb{E}^{\mathbb{P}}[M_t\mathbf{1}_A], \quad 0 \le t < T. \tag{2.10}$$

Observe that, for $0 \le t \le T$ we have $M_t = \mathbb{E}^{\mathbb{P}}[M_T \mid \mathcal{F}_t]$ (since $M_t$ is a martingale). Hence, we have, for all $A \in \mathcal{F}_t$,

$$\mathbb{Q}_t(A) = \mathbb{Q}(A). \tag{2.11}$$

We can now state the main result of this section, which is the CMG Theorem in the case of a one-dimensional standard Brownian motion.

**Theorem 2.1.6** (CMG in $\mathbb{R}$)**.** *Let $(\Omega, \mathcal{F}, P)$, $B_t$, $X_t$, $\mathcal{F}_t$, and $T$ be as above. Set:*

$$W_t := B_t - \int_0^t X_s\,ds, \quad t \in [0, T]. \tag{2.12}$$

*Then, for any fixed $T > 0$ the process $\{W_t; 0 \le t \le T\}$ is an $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$ (i.e. with respect to $\mathbb{Q}$).*

**Remark 2.1.7.** *The above construction which precedes the CMG Theorem allows us to exhibit a probability measure which is suitable for the process $W$ to remain an $\mathcal{F}_t$-Brownian motion under. In what follows, we will only need to know that such a probability measure exists, however, I thought interesting to include where this results originates from.*

## 2.2 A formal introduction to the model

In this whole section, we study the following system of SDEs. Let $N \in \mathbb{N}$. In the standard model with piecewise constant coefficients, the equations read as follows. For all $i \in \{1, ..., N\}$,

$$\mathrm{d}X_t^{i,N} = \sum_{k=1}^{N} b_k^N \mathbf{1}_{X_t^{i,N}=X_t^{(k)}} \,\mathrm{d}t + \sum_{k=1}^{N} \sigma_k^N \mathbf{1}_{X_t^{i,N}=X_t^{(k)}} \,\mathrm{d}B_t^i, \tag{$AS_N$}$$

where the $b_k^N$ are real constants, the $\sigma_k^N$ are strictly positive constants, and the $(B_t^i)_{t\geq 0}$ are independent standard Brownian motions. Furthermore, for all $t \geq 0$, we denote by

$$X_t^{(1)} \leq X_t^{(2)} \leq ... \leq X_t^{(N)}$$

the order statistics of the particle system. This model is the basis for studying rank-based interactions and was introduced in [Fer02]. As we mentioned before, such rank-based equations, though first introduced (and mostly studied) in the context of financial portfolios, find relevance in a biological setting. One can imagine that $X_t^{i,N}$ measures the weight of the $i^{\text{th}}$ in a shoal of $N$ fish at time $t$. It is then convenient to encapsulate certain phenomena by tweaking the ranked coefficients $b_N^k$ and $\sigma_n^k$. For instance, it can be observed in such a population that a certain proportion of the largest individuals, say the 1% heaviest, exhibit dominant, voracious behavior which further increases their "growth rate" $b$. The volatility of fish growth may also depend on how their weight compares to the rest of the pack, making them more or less efficient in hunting prey than their fellow individuals. Figure 3 shows a simple simulation of the classic standard model where all drift coefficients are null except for the one of the last-ranked particle, with constant volatility. One may notice that, considering measurable biological quantity such as weight, or length, which are bound to be strictly positive, we should be working with reflected processes. We did not, however, include these considerations in this work.

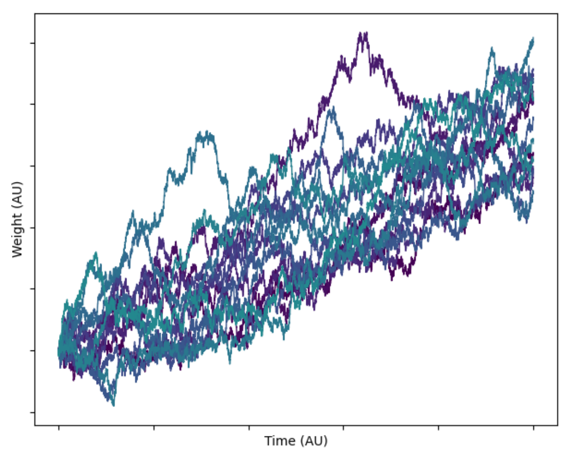


Figure 3: The Atlas model, simulated for $N = 15$ particles.

We are looking to solve the Atlas system, and we will investigate two different approaches which encapsulate standard techniques for finding both weak and strong solutions to such complex systems of SDEs. Having a good understanding of these classic methods was the first objective of my internship.

## 2.3 Strong existence and pathwise uniqueness

In this section, we will show that the system $(AS_N)$ has a strong solution, and that it is pathwise unique. This result was proven by Ichiba, Karatzas and Shkolnikov in [IKS13], who make use of the case of $N = 2$ particles which was investigated in [FIKP12]. We recall that a strong solution is defined with respect to a fixed collection of independent Brownian motions $(B^i)_{1\leq i\leq N}$ and an initial condition $X_0^{1,N} \leq ... \leq X_0^{N,N}$.

### 2.3.1 The case of two particles

First, let us consider the specific case of $N = 2$ particles, as studied by Fernholz *et al.* in [FIKP12]. The model formulates as follows:

$$\begin{cases} dX_t^1 = (b_1\mathbf{1}_{X_t^1\leq X_t^2} - b_2\mathbf{1}_{X_t^1>X_t^2})dt + (\sigma_1\mathbf{1}_{X_1^t\leq X_2^t} + \sigma_2\mathbf{1}_{X_1^t>X_2^t})dB_t^1 \\ dX_t^2 = (b_1\mathbf{1}_{X_t^1>X_t^2} - b_2\mathbf{1}_{X_t^1\leq X_t^2})dt + (\sigma_1\mathbf{1}_{X_1^t>X_2^t} + \sigma_2\mathbf{1}_{X_1^t\leq X_2^t})dB_t^1, \end{cases} \tag{2.13}$$

with initial condition $(X_0^1, X_0^2) = (x_1, x_2)$ and the following hypotheses:

(i) $B^1 \perp\!\!\!\perp B^2$ are two independent standard Brownian motions.

(ii) The constants $b_1,\ b_2 \geq 0$ and $\sigma_1, \sigma_2 \geq 0$ are such that $b_1 + b_2 > 0$ and $\sigma_1^2 + \sigma_2^2 = 1$ (at least one is nonnegative, and we normalize the diffusion coefficients).

The idea of Fernholz *et al.* is the following: we start by constructing a weak solution which is unique in distribution. This is the content of Proposition 2.3.1. We can then show that the unique weak solution is actually strong, and pathwise unique. This is addressed in Theorem 2.3.3. The question of extending these results to three or more particles will then be relevant to combinatorics rather than probability. We refer to Section 2.1 for details on weak and strong solutions of SDEs.

**Proposition 2.3.1.** *([FIKP12]) Under the assumptions above, the stochastic differential system (2.13) has a weak solution, unique in distribution.*

*Proof.* **Analysis**

We begin by introducing a weak solution to (2.13), which consists of a filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$ on which are constructed a pair of independent, $\mathcal{F}_t$-adapted, standard Brownian motions $(B^1, B^2)$, and a pair of $\mathcal{F}_t$-adapted continuous processes $(X^1, X^2)$ such that (2.13) holds with initial condition $X_0^1 = x_1$ and $X_0^2 = x_2$. Let us denote the following quantities:

$$\begin{cases} \lambda := b_1 + b_2 \\ \nu := b_1 - b_2 \end{cases} \quad \text{and} \quad \begin{cases} y := x_1 - x_2 \\ z := x_1 + x_2. \end{cases}$$

Let us now rewrite $X^1$ and $X^2$ by introducing their sum and difference:

$$\begin{cases} Y_t := X_t^1 - X_t^2 \\ Z_t := X_t^1 + X_t^2. \end{cases}$$

Defining the process $W_t := \sigma_1 W_t^1 + \sigma_2 W_t^2$, where:

$$W_t^1 := \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1 - \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2, \tag{2.14}$$

$$W_t^2 := \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^1 - \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^2, \tag{2.15}$$

we notice that $W^1$ and $W^2$ are continuous $\mathcal{F}_t$-martingales, and we can compute their quadratic variation and covariation. On the one hand:

$$\begin{aligned} \langle W^1\rangle_t = \langle W^2\rangle_t &= \left\langle \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1 \right\rangle_t + \left\langle \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2 \right\rangle_t + \left\langle \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1, \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2 \right\rangle_t \\ &= \left\langle \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1 \right\rangle_t + \left\langle \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2 \right\rangle_t \\ &= \int_0^t \mathbf{1}^2_{\{Y_s>0\}} ds + \int_0^t \mathbf{1}^2_{\{Y_s\leq 0\}} ds \\ &= t, \end{aligned}$$

where we used the independence between $B^1$ and $B^2$. On the other hand, knowing that $\mathbf{1}_{\{Y_t>0\}}\mathbf{1}_{\{Y_t\leq 0\}} = 0$ and $\langle B^1, B^2\rangle_t = 0$ for all $t \geq 0$.

$$\begin{aligned} \langle W^1, W^2\rangle_t &= \left\langle \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1 - \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2, \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^1 - \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^2, \right\rangle_t \\ &= 0. \end{aligned}$$

Hence $W^1$ and $W^2$ are two independent standard Brownian motions. Since $\sigma_1^2 + \sigma_2^2 = 1$, $W$ is a standard Brownian motion as well. We can now rewrite $Y$ as the solution of the following SDE:

$$Y_t = y - \lambda \int_0^t \operatorname{sgn}(Y_s) ds + W_t, \qquad t \geq 0, \tag{2.16}$$

where we denote sgn : $x \mapsto \mathbf{1}_{x>0} - \mathbf{1}_{x\leq 0}$. From Karatzas and Shreve [KS91, page 220], we know that the solution to this SDE is strong and pathwise unique, hence unique in distribution.

Similarly, we can define $V_t := \sigma_1 V_t^1 + \sigma_2 V_t^2$, where:

$$V_t^1 := \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^1 + \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^2, \tag{2.17}$$

$$V_t^2 := \int_0^t \mathbf{1}_{\{Y_s\leq 0\}}\, dB_s^1 + \int_0^t \mathbf{1}_{\{Y_s>0\}}\, dB_s^2. \tag{2.18}$$

It is easy to see that $V^1$, $V^2$ are also independent standard Brownian motions, and so is $V$. We can then rewrite $Z$ as:

$$Z_t = X_t^1 + X_t^2 = z + \nu t + V_t. \tag{2.19}$$

**Remark 2.3.2.** *It is important to note that rewriting $Y$ and $Z$ as in (2.16) and (2.19) is not enough to guarantee that the weak solution to (2.13) is unique in distribution. Even though $Y$, and thus $Z$ is (weakly) uniquely defined, the joint distribution of the pair $(Y, Z)$ is not necessarily unique. We can compute $X^1$ and $X^2$ as :*

$$X_t^1 = \frac{Z_t + Y_t}{2} \quad \text{and} \quad X_t^2 = \frac{Z_t - Y_t}{2}, \tag{2.20}$$

*but this is not enough to guarantee the uniqueness of the joint distribution of the pair $(X^1, X^2)$, hence of the weak solution. We have to dig deeper!*

We can notice that, given the above constructions, we can write the following intertwinements:

$$V_t^1 = \int_0^t \operatorname{sgn}(Y_s)\, dW_s^1 \quad \text{and} \quad V_t^2 = -\int_0^t \operatorname{sgn}(Y_s)\, dW_s^2, \tag{2.21}$$

as well as:

$$W_t^1 = \int_0^t \operatorname{sgn}(Y_s)\, dV_s^1 \quad \text{and} \quad W_t^2 = -\int_0^t \operatorname{sgn}(Y_s)\, dV_s^2. \tag{2.22}$$

Denoting by $\mathcal{F}^X := (\sigma(X_s, s \leq t))_{t\geq 0}$ the natural filtration generated by X, the strong existence and uniqueness of the solution of equation (2.16) yields $\mathcal{F}^Y = \mathcal{F}^W$. We shall now introduce the *skew* representations of $X^1$ and $X^2$. First of all, we need a new set of standard Brownian motions:

$$\begin{cases} \tilde{W}_t := \sigma_1 W_t^1 - \sigma_2 W_t^2 \\ \tilde{V}_t := \sigma_1 V_t^1 - \sigma_2 V_t^2, \end{cases} \tag{2.23}$$

from which we deduce new intertwinements:

$$V_t = \int_0^t \operatorname{sgn}(Y_s)\, d\tilde{W}_s \quad \text{and} \quad \tilde{V}_t = \int_0^t \operatorname{sgn}(Y_s)\, dW_s. \tag{2.24}$$

We can further decompose $\tilde{W}_t = \gamma W_t + \delta \, \tilde{U}_t$, where we define:

$$\tilde{U}_t := \sigma_2 W^1 - \sigma_1 W^2, \tag{2.25}$$

a standard Brownian motion, independent from $W$. We also denote $\gamma := \sigma_1^2 - \sigma_2^2$ and $\delta := \sqrt{1-\gamma^2} = 2\sigma_1\sigma_2$. We can now rewrite the process $V$ as follows, starting from (2.24) and :

$$\begin{aligned} V_t &= \int_0^t \operatorname{sgn}(Y_s) \left[\gamma dW_s + \delta d\tilde{U}_s\right] \\ &= \int_0^t \gamma \operatorname{sgn}(Y_s) \big[dY_s + \lambda \operatorname{sgn}(Y_s) ds\big] + \delta Q_t, \end{aligned}$$

where we denote:

$$Q_t := \int_0^t \operatorname{sgn}(Y_s) \, d\tilde{U}_s = \sigma_2 V_t^1 + \sigma_1 V_t^2. \tag{2.26}$$

The Tanaka formula 2.1.5, yields:

$$V_t = \gamma \left( |Y_t| - |y| + \lambda t - 2L_t^Y \right) + \delta Q_t, \tag{2.27}$$

where $L^Y = \left\{ L_t^Y, \, 0 \le t < \infty \right\}$ denotes the cumulative local time at the origin of the process. $Y$.

Note that $Q$ is a standard Brownian motion independent from $W$, thus from $Y$. Indeed, we immediatly get $\langle Q, W \rangle_t = \left\langle \tilde{U}, W \right\rangle_t = 0$. Let us now combine (2.19), (2.20) and (2.27) to compute a new algebraic expression of $X^1$ and $X^2$. Starting with $X^1$, we get:

$$X_t^1 = \frac{Z_t + Y_t}{2} = \frac{1}{2}(x_1 + x_2 - \gamma \, |y|) + \frac{1}{2}(Y_t + \gamma \, |Y_t|) + \frac{1}{2}(\nu + \lambda\gamma)t - \gamma L_t^Y + \frac{1}{2}\delta Q_t. \tag{2.28}$$

Introducing the positive and negative part functions, denoted by $x^+$ and $x^-$ respectively for any real number $x$, we can rewrite this expression as:

$$X_t^1 = \frac{1}{2}\delta Q_t - \gamma L_t^Y + \frac{1}{2}(\nu + \lambda\gamma)t + \frac{1}{2}(x_1 + x_2 - \gamma y^+ - \gamma y^-) + \frac{1}{2}(\gamma + 1)Y_t^+ + \frac{1}{2}(\gamma - 1)Y_t^-.$$

Given the definition of $\gamma$, it is easy to check that we have the following identity:

$$\frac{1}{2}(x_1 + x_2 - \gamma y^+ - \gamma y^-) = x_1 - \sigma_1^2 y^+ + \sigma_2^2 y^-. \tag{2.29}$$

This yields the skew representation of $X^1$:

$$X_t^1 = x_1 + \mu t + \sigma_1^2 (Y_t^+ - y^+) - \sigma_2^2 (Y_t^- - y^-) - \gamma L_t^Y + \sigma_1\sigma_2 Q_t. \tag{2.30}$$

Following the same ideas, we arrive at the skew representation of $X^2$:

$$X_t^2 = x_2 + \mu t - \sigma_2^2 (Y_t^+ - y^+) + \sigma_1^2 (Y_t^- - y^-) - \gamma L_t^Y + \sigma_1\sigma_2 Q_t. \tag{2.31}$$

From the skew representations, we immediatly deduce the uniqueness in distribution of the weak solution of (2.13). Indeed, since $Y \perp\!\!\!\perp Q$, the marginal distributions of $Y$ as the (weakly) unique solution of (2.16), and of $Q$ as a standard Brownian motion, determine the joint distribution of the pair $(Y, Q)$. From (2.30) and (2.31), this implies that the joint distribution of the pair $(X^1, X^2)$ is uniquely defined, which concludes the analysis.

**Synthesis**

We now move on to the proof of existence of a weak solution to (2.13), which we will construct using the results of our previous analysis. Let us start by introducing a filtered space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$, where $(\mathcal{F}_t)_{t\geq 0}$ is such that we can define two independent, standard Brownian motions $W^1$ and $W^2$. We can suppose, without loss of generality, that $(\mathcal{F}_t)_{t\geq 0} = \mathcal{F}^{(W^1,W^2)}$. Let $x_1$, $x_2$ be two real constants, and $b_1$, $b_2$, $\sigma_1$, $\sigma_2 \in \mathbb{R}^+$ such that:

$$b_1 + b_2 > 0 \text{ and } \sigma_1^2 + \sigma_2^2 = 1.$$

We can now construct the pairs $(W, \tilde{U})$ and $(U, \tilde{W})$ of independent standard Brownian motions using the same notations as (2.25) for $\tilde{U}$, and defining $U_t := \sigma_2 W_t^1 + \sigma_1 W_t^2$, and $Y$ as the unique strong solution of the equation (2.16), driven this time by the exogenous (*i.e.* not constructed directly from $Y$) standard Brownian motion $W$. From $Y$, we can take advantage of the intertwinements (2.21) and (2.22) in order to define $V^1$ and $V^2$ as:

$$V_t^1 = \int_0^t \operatorname{sgn}(Y_s)\, dW_s^1 \quad \text{and} \quad V_t^2 = -\int_0^t \operatorname{sgn}(Y_s)\, dW_s^2, \tag{2.32}$$

and it is easy to check that they are independent standard Brownian motions. We can then introduce two additional pairs of standard Brownian motions $(V, \tilde{Q})$ and $(Q, \tilde{V})$, where:

$$\tilde{Q}_t := \sigma_2 V_t^1 - \sigma_1 V_t^2 = \int_0^t \operatorname{sgn}(Y_s) dU_s. \tag{2.33}$$

The intertwinements from (2.24) and (2.26) are still valid, as well as the $\tilde{Q}_t = \int_0^t \operatorname{sgn}(Y_s) dU_s$. From the definitions, we also note the following filtration identity:

$$\mathcal{F}^{(V^1,V^2)} = \mathcal{F}^{(V,\tilde{Q})} = \mathcal{F}^{(\tilde{V},Q)}. \tag{2.34}$$

Let us now define two continuous, $\mathcal{F}_t$-martingales, as:

$$M_t^1 = \int_0^t \left(\sigma_1 \mathbf{1}_{\{Y_s>0\}}\, dW_s^1 + \sigma_2 \mathbf{1}_{\{Y_s\leq 0\}}\, dW_s^2\right), \tag{2.35}$$

$$M_t^2 = \int_0^t \left(-\sigma_1 \mathbf{1}_{\{Y_s\leq 0\}}\, dW_s^1 - \sigma_2 \mathbf{1}_{\{Y_s>0\}}\, dW_s^2\right). \tag{2.36}$$

We can finally introduce $X^1$ and $X^2$ as:

$$X_t^1 = x_1 + \int_0^t \left(b_1 \mathbf{1}_{\{Y_s\leq 0\}} - b_2 \mathbf{1}_{\{Y_s>0\}}\right) ds + M_t^1, \tag{2.37}$$

$$X_t^2 = x_2 + \int_0^t \left( b_1 \mathbf{1}_{\{Y_s>0\}} - b_2 \mathbf{1}_{\{Y_s \le 0\}} \right) ds + M_t^2. \tag{2.38}$$

It is now easy to check that, according to our construction, we have:

$$X_t^1 - X_t^2 = Y_t \ \text{ and } \ X_t^1 + X_t^2 = x_1 + x_2 + \nu t + V_t, \tag{2.39}$$

where $\nu$ is defined as in the previous section. We can also check that $\langle M^1, M^2 \rangle_t$=0 for all $t \geq 0$, and compute the quadratic variations:

$$\left\langle M^1 \right\rangle_t = \int_0^t (\sigma_1^2 \mathbf{1}_{Y_s>0} + \sigma_2^2 \mathbf{1}_{Y_s \le 0}) ds,$$

$$\left\langle M^2 \right\rangle_t = \int_0^t (\sigma_1^2 \mathbf{1}_{Y_s \le 0} + \sigma_2^2 \mathbf{1}_{Y_s>0}) ds.$$

From the martingale representation theorem (see [KS91]), we know that there exist two independent Brownian motions $B^1$ and $B^2$ on our filtration such that:

$$M_t^1 = \int_0^t \left( \sigma_1 \mathbf{1}_{\{Y_s>0\}} + \sigma_2 \mathbf{1}_{\{Y_s \le 0\}} \right) dB_s^1, \tag{2.40}$$

$$M_t^2 = \int_0^t \left( \sigma_1 \mathbf{1}_{\{Y_s \le 0\}} + \sigma_2 \mathbf{1}_{\{Y_s>0\}} \right) dB_s^2. \tag{2.41}$$

It is then straightforward to reformulate $X^1$ and $X^2$ and cast them in the form (2.13), of which we thus have constructed a weak solution, unique in distribution.

□

We now turn to the main result of [FIKP12], which recovers the strength and pathwise uniqueness of the solution in the two-particle system.

**Theorem 2.3.3.** *Under the same assumptions, the weak solution constructed in Proposition 2.3.1 is actually strong, and pathwise unique.*

*Proof.* We will only consider the non-degenerate case where $\sigma_1 \sigma_2 > 0$. The degenerate case, which is also considered in [FIKP12], does not allow to show existence nor uniqueness in the case of $N > 2$ particles.

It is enough to show that the following holds:

$$\mathcal{F}^{(X^1,X^2)} \subseteq \mathcal{F}^{(B^1,B^2)}, \tag{2.42}$$

*i.e.* that $(X^1, X^2)$ is a measurable function of the pair of Brownian motions $(B^1, B^2)$. First of all, let us show the following equality:

$$\mathcal{F}^{(X^1,X^2)} = \mathcal{F}^{(W^1,W^2)}. \tag{2.43}$$

The first inclusion relation $\mathcal{F}^{(X^1,X^2)} \subseteq \mathcal{F}^{(W^1,W^2)}$ is easily obtained from (2.35)-(2.38), by construction. Furthermore, from (2.35) and (2.36), $M^1$ and $M^2$ are $\mathcal{F}^{(W^1,W^2)}$-martingales, but also $\mathcal{F}^{(X^1,X^2)}$-martingales. Indeed, since $\mathcal{F}^{(X^1,X^2)} \subseteq \mathcal{F}^{(W^1,W^2)}$, the tower property of conditional expectations yields for $M^1$ (resp. $M^2$):

$$\begin{aligned}\forall s < t, \quad \mathbb{E}\left[M_t^1 \mid \mathcal{F}_s^{(X^1,X^2)}\right] &= \mathbb{E}\left[\mathbb{E}\left[M_t^1 \mid \mathcal{F}_s^{(W^1,W^2)}\right] \mid \mathcal{F}_s^{(X^1,X^2)}\right] \\ &= \mathbb{E}\left[M_s^1 \mid \mathcal{F}_s^{(X^1,X^2)}\right] \\ &= M_s^1,\end{aligned}$$

where we get the last equality from (2.37) and (2.38) which show that $M^1$ and $M^2$ are $\mathcal{F}_s^{(X^1,X^2)}$-measurable, and (2.43) then follows.

Let us now treat the equal variance case separately. Suppose that $\sigma_1^2 = \sigma_2^2 = \frac{1}{2}$. We can easily check that in this scenario:

$$W_t = \frac{B_t^1 - B_t^2}{\sqrt{2}},$$

which yields $\mathcal{F}^W \subseteq \mathcal{F}^{(B^1,B^2)}$. Since $Y$ is strongly defined, we also have $\mathcal{F}^Y = \mathcal{F}^W$, hence:

$$\mathcal{F}^{(X^1,X^2)} = \mathcal{F}^{(W^1,W^2)} \subseteq \mathcal{F}^{(B^1,B^2)}. \tag{2.44}$$

According to Yamada and Watanabe, strong existence and pathwise uniqueness are equivalent when uniqueness in distribution already holds. This proves the existence of a pathwise unique, strong solution to (2.13) in the case of equal variances.

The case of different variances, though similar, involves a change of probability measure, but it is quite straightforward, and we will not go into further details for the sake of conciseness.

□

### 2.3.2 Extending to a three-particle system

We now turn our attention to the work of Ichiba *et al.* [IKS13]. The first step in generalizing Proposition 2.3.1 is to extend the result for $N = 3$ particles. As we will see, the proof of this extension, as well as the generalization to a finite particle system, relies on ideas of combinatorics rather than probability theory. For this reason, we will only present the outline of both proofs.

The model formulates as follows. For $i = 1, 2, 3$,

$$\mathrm{d}X_t^i = \sum_{k=1}^{3} b_k \mathbf{1}_{X_t^i = X_t^{(k)}} \,\mathrm{d}t + \sum_{k=1}^{3} \sigma_k \mathbf{1}_{X_t^i = X_t^{(k)}} \,\mathrm{d}B_t^i, \tag{2.45}$$

with an initial condition which satisfies $X_0^1 < X_0^2 < X_0^3$. Let $\tau$ be the first time of triple collision in the particle system, defined as:

$$\tau := \inf\left\{t \geq 0 : \, X_t^1 = X_t^2 = X_t^3\right\}.$$

We can now extend Proposition 2.3.1.

**Proposition 2.3.4.** *We have the following two cases:*

*(i) If $\sigma_2^2 - \sigma_1^2 \leq \sigma_3^2 - \sigma_2^2$, then the system (2.45) has a strong, pathwise unique solution.*

*(ii) Else, if $\sigma_2^2 - \sigma_1^2 > \sigma_3^2 - \sigma_2^2$, then the strong solution still exists, and is pathwise unique until time $\tau$.*

**Remark 2.3.5.** *The above condition used to distinguish the two cases is more general, and gives a sufficient condition for the absence, a.s., of triple collision in the particle system. It will be properly introduced later.*

*Proof.* The idea for the proof of this result is to piece together path segments consisting of a two-particle system in interaction and a third, independent one up until there is a collision with this third particle. The difficulty resides in the well-definition of the associated stopping times. For all closed time intervals $[a, b]$, we denote by:

$$Z^{[a,b],B,W}(b_1, b_2, c_1, c_2) \tag{2.46}$$

the strong solution of the two-particle system, which we constructed earlier, with Brownian motions $B, W$, and coefficients $b_1, b_2, c_1, c_2$. We can now introduce the sequence of stopping times:

$$0 = \tau_0 \leq \rho_0 \leq \tau_1 \leq \rho_1 \leq \cdots , \tag{2.47}$$

and construct the strong solution to the three-particle system inductively, on the intervals $[\tau_k, \rho_k]$, $[\rho_k, \tau_{k+1}]$, $k \geq 0$:

$$X^{\pi_k(1)}\big([\tau_k, \rho_k]\big) := \big(Z^{[\tau_k,\rho_k],\, W^{\pi_k(1)},\, W^{\pi_k(2)}}(\delta_1, \delta_2, \sigma_1, \sigma_2)\big)_1, \tag{2.48}$$

$$X^{\pi_k(2)}\big([\tau_k, \rho_k]\big) := \big(Z^{[\tau_k,\rho_k],\, W^{\pi_k(1)},\, W^{\pi_k(2)}}(\delta_1, \delta_2, \sigma_1, \sigma_2)\big)_2, \tag{2.49}$$

$$X^{\pi_k(3)}(t) := X^{\pi_k(3)}(\tau_k) + \delta_3\,(t - \tau_k) + \sigma_3\big(W^{\pi_k(3)}(t) - W^{\pi_k(3)}(\tau_k)\big), \qquad t \in [\tau_k, \rho_k], \tag{2.50}$$

and $\rho_k$, which is defined by:

$$\rho_k := \inf\big\{t > \tau_k :\ X^{\pi_k(3)}(t) = X^{\pi_k(2)}(t) \ \text{ or } \ X^{\pi_k(3)}(t) = X^{\pi_k(1)}(t)\big\}, \tag{2.51}$$

is the first instant of collision of the "independent" particle with one of the two remaining, interacting particles. After the collision at time $\rho_k$, the independent particle becomes part of the new interacting two-particle system. Hence, the system now writes:

$$X^{\theta_k(1)}(t) := X^{\theta_k(1)}(\rho_k) + \delta_1\,(t - \rho_k) + \sigma_1\big(W^{\theta_k(1)}(t) - W^{\theta_k(1)}(\rho_k)\big), \qquad t \in [\rho_k, \tau_{k+1}], \tag{2.52}$$

$$X^{\theta_k(2)}\big([\rho_k, \tau_{k+1}]\big) := \big(Z^{[\rho_k,\tau_{k+1}],\, W^{\theta_k(1)},\, W^{\theta_k(2)}}(\delta_2, \delta_3, \sigma_2, \sigma_3)\big)_1, \tag{2.53}$$

$$X^{\theta_k(3)}\big([\rho_k, \tau_{k+1}]\big) := \big(Z^{[\rho_k,\tau_{k+1}],\, W^{\theta_k(1)},\, W^{\theta_k(2)}}(\delta_2, \delta_3, \sigma_2, \sigma_3)\big)_2, \tag{2.54}$$

and we define $\tau_{k+1} := \inf\{t > \rho_k : \; X^{\theta_k(2)}(t) = X^{\theta_k(1)}(t) \;\text{ or }\; X^{\theta_k(3)}(t) = X^{\theta_k(1)}(t)\}$. For each $k \geq 0$, we have denoted here by $\pi_k$ a permutation of the set $\{1, 2, 3\}$ such that

$$X^{\pi_k(1)}(\tau_k) \leq X^{\pi_k(2)}(\tau_k) \leq X^{\pi_k(3)}(\tau_k),$$

and by $\theta_k$ a permutation of the set $\{1, 2, 3\}$ such that

$$X^{\theta_k(1)}(\rho_k) \leq X^{\theta_k(2)}(\rho_k) \leq X^{\theta_k(3)}(\rho_k).$$

It is then quite straightforward to check that the construction is coherent, and considerations on the limit of these sequences of stopping times yield the existence. The pathwise uniqueness, on the other hand, is proven by using results from [Che01]. □

### 2.3.3 The finite particle system

We now move on to the general result, which is the central theorem in [IKS13]. The result is stated as follows:

**Theorem 2.3.6.** *Let $\tau$ be the first time of triple collision in the particle system, defined as:*

$$\tau := \inf\{t \geq 0, \; \exists i, j, k, \; X_t^{i,N} = X_t^{j,N} = X_t^{k,N}\}.$$

*Then the system $(AS_N)$ has a unique, strong solution, defined up to time $\tau$.*

**Remark 2.3.7.** *Note that we know a sufficient condition for there to be no triple collision in the system, almost surely. If the sequence $(0, \sigma_1^2, \sigma_2^2, ..., \sigma_N^2, 0)$ is concave, that is, for any three consecutive elements of the sequence, say $\sigma_i^2, \sigma_{i+1}^2, \sigma_{i+2}^2$, we have:*

$$\sigma_{i+1}^2 \geq \frac{1}{2}(\sigma_{i+2}^2 + \sigma_i^2), \tag{2.55}$$

*then $\tau = \infty$ almost surely.*

We now present a few elements of proof for Theorem 2.3.6.

*Proof.* The idea behind the proof of this result is similar to the previous one. By introducing the right sequence of stopping times, and taking its limit, we construct a strong solution up to a certain stopping time $\tau_0^{[1]}$. Since, for $N > 3$, it is not immediate that $\tau_0^{[1]} = \tau$, we need to keep going! Assuming that $\tau_0^{[1]} \neq \tau$, we can keep constructing the solution on a second level of induction, taking the limit of the sequence of stopping times $(\tau_k^{[1]})_{k\geq 0}$. The limit $\tau^{[\infty]}$ is still not necessarily equal to $\tau$. Adding a third level of induction allows us to guarantee that strictly more than half of the particles of the system collide with another particle at a common stopping time $\tilde{\tau}$, meaning that there is at least one triple collision at that instant. We have thus constructed a strong solution of the system up until time $\tau$. Pathwise uniqueness is then shown in the same way as for the three-particle system. □

## 2.4 Weak existence and uniqueness via a martingale problem

In this section, we present the results of [BP87], in which the authors show that diffusion equations with piecewise constant coefficients have a unique, weak solution under the right hypotheses. The authors do so by introducing the expected generator of the solution of an SDE with piecewise constant coefficients. The generator of such a process, if it exists, is directly linked to the weak solutions of the corresponding SDE, through what is called a martingale problem. This approach differs completely from the "hands-on" method presented [IKS13] and [FIKP12] for crafting a solution. The arguments here are more subtle, and worth presenting although we already constructed a unique weak (and even strong) solution. For the sake of conciseness, we will only present the main ideas of the proofs of [BP87].

### 2.4.1 The martingale problem

Consider the following operator, for all $f \in C^2(\mathbb{R}^N)$:

$$L_{a,b}f : x \mapsto \frac{1}{2}\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{N} b_i(x)\frac{\partial f}{\partial x_i}(x), \tag{2.56}$$

where the $a_{i,j}, b_i$ are bounded, measurable functions, and the matrix $a := (a_{i,j})$ is uniformly positive definite. Assume that the space $\mathbb{R}^N$ can be partitioned into finitely many polyhedra, the interior of which $a$ is constant in. In order to show that the operator $L_{a,b}$ is the generator of a unique strong Markov process, Stroock and Varadhan in [SV79] showed that it is strictly equivalent to showing that there is a unique solution to the following martingale problem:

*For each initial condition $x_0$, there is a unique probability measure $P$ on $C([0,\infty), \mathbb{R}^N)$ such that:*

*(i)* $P[X_0 = x_0] = 1$;

*(ii) For all* $f \in C^2(\mathbb{R}^N)$, $f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,\mathrm{d}s$ *is a $P$-local martingale.*

### 2.4.2 Existence and uniqueness of a weak solution

We know that, when the $a_{ij}$ are continuous on $\mathbb{R}^N$, the results of Stroock and Varadhan imply that there is a unique solution to the martingale problem, and thus a unique strong Markov process associated to $L_{a,b}$. The main result of [BP87] extends this result in the case of piecewise constant coefficients under regularity conditions. It reads as follows:

**Theorem 2.4.1.** *Assume that $a$ is measurable, uniformly bounded and uniformly positive definite. Moreover, suppose that $b$ is measurable and locally bounded with at most spatially linear growth. Let $x_0\mathbb{R}^N$, and assume that $\mathbb{R}^N$ can be divided into finitely many polyhedra such that $a$ is constant in the interior of each polyhedron. Then, the martingale problem for $L_{a,b}$ has a unique solution starting at $x_0$.*

*Proof.* Let us give indications on the main ideas behind the proof of Theorem 2.4.1. The parts of $\mathbb{R}^N$ which need work are the boundary points of the polyhedra which split the whole space. The proof is split into two parts: one which deals with nonvertex boundary points, and the second which deals with vertex boundary points. Let $A_1, ..., A_n$ be a collection of polyhedra such that:

$$\mathbb{R}^N = \bigcup_{1 \leq i \leq n} \bar{A}_i, \tag{2.57}$$

and the $A_i$ have disjoint interiors. On the one hand, $x \in \bigcup_{i=1}^n \partial A_i$ is a nonvertex boundary point if, up to a change of coordinates, in a neighborhood of $x$, $a$ depends on strictly less than $N$ coordinates of its variable. Otherwise, $x$ is called a boundary point.

First, let us consider nonvertex boundary points. We outline this part of the proof by presenting the intermediate results used.

**Proposition 2.4.2.** *Suppose that the solution to the martingale problem for the $k \times k$ matrix $a$ starting from $y_0$ is unique, for a certain $k < N$. Then the solution to the martingale problem starting from $(y_0, z_0) \in \mathbb{R}^N$ for $\tilde{a}$ is also unique, where $\tilde{a}$ is the extension of $a$ to the dimension $N \times N$ using the $(N-k) \times (N-k)$ identity matrix.*

**Lemma 2.4.3.** *Suppose that $a$ is an $N \times N$ positive definite matrix, and let $k < N$. Then, there exists an $N \times N$ matrix of the form $\sigma = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix}$, where $A, C$ are positive definite, of dimensions $k \times k$ and $(N-k) \times (N-k)$ respectively, and $\sigma\sigma^T = a$.*

These two results are useful in proving the following theorem, which is the main result of the first part of the global proof.

**Theorem 2.4.4.** *Let $k < N$ and suppose that $a(x)$ depends only on the first $k$ coordinates of $x$. Suppose that $a = \begin{pmatrix} D & F^T \\ F & G \end{pmatrix}$, where $D$ is of dimension $k \times k$, and suppose that the solution to the martingale problem starting from $y_0$ for $D$ is unique. Then the solution to the martingale problem starting from $(y_0, z_0)$ is unique for all $z_0 \in \mathbb{R}^{N-k}$.*

The second part of the proof, in which we consider vertex boundary points, relies on the use of the Krein-Rutman theorem, which is a generalization of the Perron-Frobenius in infinite-dimensional Banach spaces (see [KR62]). Then, there is remaining work to properly show uniqueness of the solution by piecing together the results obtained for nonvertex and vertex boundary points. We will not go into further details since, although studying Bass and Pardoux's work was part of my internship, it is not the main focus of this report.

$\square$

# 3 Propagation of chaos for a rank-based process

## 3.1 Kac's propagation of chaos

The idea behind propagation of chaos is that, in a sufficiently smooth particle system, the interactions between given particles become negligible as the total number of particles grows to infinity. As a result, in any finite-sized subset of the population, the particles become independent and identically distributed. Let us give a more formal introduction to the essential concepts behind Kac's description of propagation of chaos, introduced in [Kac56]. Consider the following real-valued particle system:

$$X_t^N = (X_t^{1,N}, ...X_t^{N,N}) \sim f_t^N.$$

Suppose that the system is exchangeable, *i.e.*, that $f_t^N$ is symmetric under any permutation of the particles.

**Definition 3.1.1** (Chaos)**.** *Let $f \in \mathcal{P}(\mathbb{R})$. A sequence of symmetric probability measures $\left(f^N\right)_{N\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ is said to be f-chaotic if, and only if, for all $k \in \mathbb{N}$ and $\phi_k \in C_b(\mathbb{R}^k)$:*

$$\lim_{N\to\infty} \left\langle f^N, \phi_k \otimes \mathbf{1}^{\otimes N-k} \right\rangle = \left\langle f^{\otimes k}, \phi_k \right\rangle, \tag{3.1}$$

*or, equivalently, $\forall k \in \mathbb{N}$, $f^{k,N} \xrightarrow{w} f^{\otimes k}$ w.r.t. the topology of weak convergence, where $f^{k,N}$ denotes the $k$-th marginal of $f^N$.*

We can now properly define the notion of chaos propagation. The idea is that, in a particle system where $\left(f_0^N\right)$ is $f_0$-chaotic, we want to show, in some sense, that this initial chaos spreads to future times $t \geq 0$. We present two definitions for propagation of chaos.

**Definition 3.1.2** (Pointwise propagation of chaos)**.** *We say that pointwise propagation of chaos holds for $(f_t)_{t\geq 0} \in C(\mathbb{R}^+, \mathcal{P}(\mathbb{R}))$ if, and only if, for all $t \geq 0$, $(f_t^N)_{N\in\mathbb{N}}$ is $f_t$-chaotic.*

**Definition 3.1.3** (Pathwise propagation of chaos)**.** *We say that pathwise propagation of chaos holds for $f \in \mathcal{P}(\mathcal{D}(\mathbb{R}^+, \mathbb{R}))$ if and only if the sequence $\left(f^N\right)_{N\in\mathbb{N}}$ is $f$-chaotic.*

**Remark 3.1.4.** *Pathwise propagation of chaos is more general, and implies its pointwise counterpart. In both cases, propagation of chaos implies that, as $N \to \infty$, any fixed-sized subset of particles behaves as an i.i.d. particle system of marginal distributions $f_t$.*

These definitions are quite straightforward. However, in [JR13], the authors rather work with the empirical distribution $\mu^N$ of the system in order to prove chaos propagation. One can find the link between Kac's definitions of chaos and the empirical measure associated with the particle system in the extensive review work by Chaintron and Diez [CD22, Lemma 3.19]. The lemma, proven by Sznitman in [Szn91], reads as follows:

**Lemma 3.1.5.** *The sequence of distributions $\left(f^N\right)_{N\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ of the particle systems $(X^N)_{N\in\mathbb{N}}$ is f-chaotic if, and only if, the empirical distribution $\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}$ of the system converges in distribution to the deterministic measure $f$.*

We are thus looking to prove the convergence of the sequence of random empirical measures $(\mu^N)_{N\in\mathbb{N}}$. In this context, the standard approach is the following. On the one hand, we must show that the sequence of distributions $(\tilde{\pi}_N)_{N\in\mathbb{N}}$ is tight in order to have relative compactness in the space of probability distributions over right-continuous paths with left limits (càdlàg). On the other hand, by showing that any limit of a converging subsequence of $(\tilde{\pi}_N)_{N\in\mathbb{N}}$ is a weak solution to a certain Cauchy problem, and that this problem only has one such solution, one effectively shows the convergence of $(\mu^N)_{N\in\mathbb{N}}$ in distribution. Let us start by presenting proofs for these results in the case of a classic Atlas model, as in [JR13]. We shall then take a swing at extending them for a generalized rank-based process with unbounded drift coefficients.

## 3.2 The case of a standard Atlas model

In this section, we present the results of chaos propagation in the Atlas model with piecewise constant coefficients. This question was investigated in [JR13], with a strong focus on the link between the asymptotic behavior of a large particle system and the existence of a unique weak solution to a certain Cauchy problem. By further showing that the sequence of distributions of the empirical measure associated with the particle system is tight, Jourdain and Reygner were able to prove propagation of chaos under a few nondegeneracy conditions.

### 3.2.1 Model formulation and the limiting SDE

Recall the standard particle system: for all $i \in \{1, ..., N\}$,

$$\mathrm{d}X_t^{i,N} = \sum_{k=1}^{N} b_k^N \mathbf{1}_{X_t^{i,N}=X_t^{(k)}}\,\mathrm{d}t + \sum_{k=1}^{N} \sigma_k^N \mathbf{1}_{X_t^{i,N}=X_t^{(k)}}\,\mathrm{d}B_t^i.$$

Let us now introduce $b, \sigma$, real-valued, continuous functions defined on $[0,1]$ such that $\sigma$ is nonnegative on $[0,1]$, and that:

$$\forall k \leq N, \quad b_k^N = b\left(\frac{k}{N}\right), \quad \sigma_k^N = \sigma\left(\frac{k}{N}\right).$$

The continuity condition ensures that, as the population size grows, the dynamics of a given subset of closely ranked particles are asymptotically identical. The empirical measure associated with the N-particle system is defined as follows: for all $N \in \mathbb{N}$,

$$\mu^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}}.$$

The empirical distribution is to be seen as a random measure over the space of càdlàg paths, *i.e.*, for all $\omega$,

$$\mu^N(\omega) = \frac{1}{N}\sum_{i=1}^{N} \delta_{X^{i,N}(\omega)} = \left(\frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^{i,N}(\omega)}\right)_{t\geq 0},$$

is itself a probability measure over the space $\mathcal{D}(\mathbb{R}_+, \mathbb{R})$. Let us now rewrite the system of SDEs in terms of the empirical distribution. For all $i \in \{1, ..., N\}$,

$$\mathrm{d}X_t^{i,N} = b\left(H * \mu_s^N(X_s^{i,N})\right)\,\mathrm{d}t + \sigma(H * \mu_s^N(X_s^{i,N}))\,\mathrm{d}B_t^i, \tag{3.2}$$

where $H : x \mapsto \mathbf{1}_{x>0} + \frac{1}{2}\delta_0(x)$ denotes the Heaviside step function, and $\mu_s^N$ is the empirical measure associated with the system at time $s \geq 0$. This formulation follows from a simple calculation: for all $s \geq 0$ and $i \in \{1, ..., N\}$),

$$\begin{aligned} H * \mu_s^N(X_s^{i,N}) &= \int_{-\infty}^{\infty} H(X_s^{i,N} - x)\mu_s^N(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{X_s^{i,N}} \mu_s^N(x)\,\mathrm{d}x \\ &= \frac{1}{N}\sum_{k=1}^{N}\int_{-\infty}^{X_s^{i,N}} \delta_{X_s^{k,N}}(x)\,\mathrm{d}x \\ &= \frac{\mathrm{rank}(X_s^{i,N})}{N}. \end{aligned}$$

We immediately retrieve the formulation from $(AS_N)$.

We are looking for a limit to the empirical measure as a collection of probability measures $(P_t)_{t\geq 0}$. By assuming its existence, and taking the formal limit in (3.2), we can write the limiting nonlinear SDE which governs the dynamics of a single particle in the infinite population asymptotics:

$$\begin{cases} X_t = X_0 + \int_0^t b(H * P_s(X_s))\,\mathrm{d}s + \int_0^t \sigma(H * P_s(X_s))dB_s \\ X_t \sim P_t, \end{cases} \tag{3.3}$$

with initial condition $X_0 \sim P_0$ given and independent from $B$.

**Remark 3.2.1.** *By nonlinear, we refer in this context to the fact that the coefficients of the SDE (3.3) depend explicitly on both $X_s$ and its distribution $P_s$. This is called nonlinearity in McKean's sense.*

Let us note that for all $s \geq 0$, $F_s : x \mapsto H * P_s(x)$ is exactly the cumulative distribution function (CDF) associated to the distribution $P_s$. We can thus rewrite the limiting SDE as follows:

$$\begin{cases} X_t = X_0 + \int_0^t b(F_s(X_s))\,\mathrm{d}s + \int_0^t \sigma(F_s(X_s)) dB_s, \\ F_t \text{ is the CDF of } X_t. \end{cases} \tag{3.4}$$

### 3.2.2 Deterministic dynamics and the Cauchy problem

We are now interested in finding a deterministic description of the dynamics of $\{F_t\}_{t\geq 0}$ as a collection of CDFs. Let us introduce a smooth test function $f \in C_c^\infty(\mathbb{R})$. Itô's formula yields, for all $t \geq 0$:

$$f(X_t) = f(X_0) + M_t + \frac{1}{2}\int_0^t \sigma^2(F_s(X_s)) f''(X_s)\,\mathrm{d}s + \int_0^t b(F_s(X_s)) f'(X_s)\,\mathrm{d}s, \tag{3.5}$$

where $(M_t)_{t\geq 0}$ is a martingale with zero expectation. By taking the expectation, and assuming that $P_t$ has density $p_t$ on $\mathbb{R}$, one gets:

$$\begin{aligned} \int_{\mathbb{R}} f(x) p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x) p_0(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_0^t \sigma^2(F_s(x)) f''(x) p_s(x)\,\mathrm{d}s\,\mathrm{d}x \\ + \int_{\mathbb{R}}\int_0^t b(F_s(x)) f'(x) p_s(x)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{3.6}$$

We can now integrate by parts both terms on the right, using the fact that $f$ has compact support. On the one hand:

$$\int_{\mathbb{R}}\int_0^t b(F_s(x)) f'(x) p_s(x)\,\mathrm{d}s\,\mathrm{d}x = -\int_{\mathbb{R}}\int_0^t f(x)\,\partial_x\left[b(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x.$$

And on the other hand,

$$\int_{\mathbb{R}}\int_0^t \sigma^2(F_s(x)) f''(x) p_s(x)\,\mathrm{d}s\,\mathrm{d}x = \int_{\mathbb{R}}\int_0^t f(x)\,\partial_x^2\left[\sigma^2(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x.$$

Combining these terms yields the following equation:

$$\begin{aligned} \int_{\mathbb{R}} f(x) p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x) p_0(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_0^t f(x)\,\partial_x^2\left[\sigma^2(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x \\ - \int_{\mathbb{R}}\int_0^t f(x)\,\partial_x\left[b(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x, \end{aligned} \tag{3.7}$$

which is exactly the weak formulation of the following nonlinear Fokker-Planck equation:

$$\partial_t p_t(x) = \frac{1}{2}\partial_x^2\left[\sigma^2(F_t(x))p_t(x)\right] - \partial_x\left[b(F_s(x))p_t(x)\right]. \tag{3.8}$$

Finally, in order to retrieve the Cauchy problem we will be studying in the following, we integrate over space, knowing that for all $t \geq 0$, $\partial_x F_t(.) = p_t(.)$. This yields:

$$\begin{cases} \partial_t F_t(x) = \frac{1}{2}\partial_x^2 A(F_t(x)) - \partial_x B(F_t(x)) \\ F_0(x) = H * P_0(x), \end{cases} \tag{3.9}$$

where $P_0$ is an arbitrary probability measure, and we defined:

$$\begin{cases} A(u) := \int_0^u \sigma^2(v)dv \\ B(u) := \int_0^u b(v)dv. \end{cases}$$

From this expression of the Cauchy problem, we easily retrieve its weak formulation as before by introducing a test function $g \in C_c^\infty([0,t] \times \mathbb{R})$. After integrating by parts over time on the left-hand side, and over space on the right-hand side:

$$\begin{aligned} &\int_{\mathbb{R}} g(t,x)F_t(x)\,\mathrm{d}x - \int_{\mathbb{R}} g(0,x)H * P_0(x)\,\mathrm{d}x \\ &\qquad = \int_{\mathbb{R}}\int_0^t \left[F_s(x)\,\partial_s g(s,x) + \frac{1}{2}A(F_s(x))\,\partial_x^2 g(s,x) + B(F_s(x))\,\partial_x g(s,x)\right]\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{3.10}$$

### 3.2.3 Propagation of chaos in the Atlas model

Let us first introduce, for all $T \in [0,\infty]$, the set $\mathcal{P}(T)$ of all continuous mappings $t \mapsto P_t$ such that for all $t < T$, $\int_{\mathbb{R}} x\, P_t(\mathrm{d}x) < \infty$, *i.e.* $P_t$ has a finite first order moment, and that the function $t \mapsto \int_{\mathbb{R}} |x| P_t(\mathrm{d}x)$ is locally integrable on $[0,T)$. Then, let us define:

$$\mathcal{F}(T) := \{F : (t,x) \mapsto (H * P_t(x);\ P \in \mathcal{P}(T)\}.$$

Note that $\mathcal{F}(T) \subset C([0,T), L^1_{\text{loc}}(\mathbb{R}))$. We denote $\mathcal{F}(\infty)$ and $\mathcal{P}(\infty)$ by $\mathcal{F}$ and $\mathcal{P}$, respectively. $\mathcal{F}$ is, in simple terms, the set of all smooth collections of CDFs indexed by $\mathbb{R}^+$.

We know from [BP87], or from [IKS13], that for any $N \geq 1$, the system of SDEs (3.2) has a weak solution $X^N = \left(X_t^{1,N}, ..., X_t^{N,N}\right)_{t\geq 0}$, unique in distribution, with empirical measure:

$$\mu^N := \frac{1}{N}\sum_{i=1}^N \delta_{X^{i,N}}.$$

We now move on to stating the main result of chaos propagation for the standard Atlas model, and giving the main elements of its proof which we gave an outline of earlier. This theorem is formulated in [JR13, Proposition 2.1].

**Theorem 3.2.2** (Propagation of chaos)**.** *Assume that the function $A$ is increasing, and that the initial distribution $P_0$ satisfies:*

$$\int_{\mathbb{R}} x\, P_0(\,\mathrm{d}x) < \infty.$$

*Then, the Cauchy problem (3.9) has a unique, weak solution in $\mathcal{F}$, and it writes $F : (t,x) \mapsto H * P_t(x)$, where $t \mapsto P_t$ is the limit in $C([0,\infty), \mathcal{P}(\mathbb{R}))$, in probability, of the sequence of mappings $t \mapsto \mu_t^N$.*

### 3.2.4 Intermediate results and proofs

In order to prove Theorem 3.2.2, we must introduce several intermediate results. Let us begin with considerations on the sequence $\left(\pi^N\right)_{N\geq 1}$ of distributions of the random mappings $t \mapsto \mu_t^N$ that is, in $C([0,\infty), \mathcal{P}(\mathbb{R})))$. Moreover, let $\left(\tilde{\pi}^N\right)_{N\geq 1}$ denote the sequence of distributions of $\mu^N$ as a random measure with values in $\mathcal{P}(C([0,\infty), \mathbb{R}))$.

**Lemma 3.2.3.** *The sequence $\left(\tilde{\pi}^N\right)_{N\geq 1}$ is tight.*

*Proof.* Recall that tightness in this context refers to the fact that for all $\epsilon > 0$, there exists a compact subset $K$ of $\mathcal{P}(C([0,\infty), \mathbb{R}))$ such that for all $N \geq 1$, $\tilde{\pi}^N(K) \geq 1 - \epsilon$. This definition can be found in [Bil99].

Following [Szn91, Proposition 2.2], since the distribution of $X^N$ is symmetric in $C\big([0,\infty), \mathbb{R}^N\big)$, $\left(\tilde{\pi}^N\right)_{N\geq 1}$ is tight if, and only if the sequence of distributions of the first coordinates $\left(X^{1,N}\right)_{N\geq 1}$ is tight. This is true according to the Kolmogorov criterion, which can also be found in [Bil99], since:

(i) The marginals $X_0^{1,N} \sim P_0$ are uniformly tight;

(ii) For all $N \geq 1$ and $s, t \geq 0$, the moments of $\left|X_t^{1,N} - X_s^{1,N}\right|$ are bounded,

and the latter is immediate since the coefficients $\sigma$ and $b$ are bounded. □

Since the canonical application $\mathcal{P}(C([0,\infty), \mathbb{R})) \to C([0,\infty), \mathcal{P}(\mathbb{R}))$ is continuous, then the tightness of the sequence $\left(\tilde{\pi}^N\right)_{N\geq 1}$ implies that of the sequence $\left(\pi^N\right)_{N\geq 1}$ of distributions of the random mappings $t \mapsto \mu_t^N$.

Now, let $\pi^\infty$ be the limit of a converging subsequence of $\left(\pi^N\right)_{N\geq 1}$, which we will still index by $N$ for clarity purposes. We wish to show that $\pi^\infty$ is uniquely defined, thus showing the convergence in distribution of our sequence.

**Lemma 3.2.4.** *Assume that the initial distribution $P_0$ verifies:*

$$\int_{\mathbb{R}} x\, P_0(\,\mathrm{d}x) < \infty.$$

*Then the distribution $\pi^\infty$ is concentrated on the set of mappings $P \in \mathcal{P}$ such that $(t,x) \mapsto H * P_t(x)$ is a weak solution to the Cauchy problem (3.9).*

*Proof.* We will prove that $\pi^\infty$ is concentrated on $\mathcal{P}$. The rest of the proof, which deals with showing that a random variable $\mu^\infty \sim \pi^\infty$ is almost surely a weak solution to (3.9), can be adapted from [Jou00]. We did not insist on this part, since we were not able to adapt this result in our rank-based Ornstein-Uhlenbeck model.

Let $\mu^\infty \sim \pi^\infty$. We need to show that, for all $t \geq 0$:

$$\sup_{s\in[0,t]} \int_{\mathbb{R}} |x|\, \mu_s^\infty(\,\mathrm{d}x) < \infty,$$

almost surely, so that, taking $t$ in a countable, unbounded subset of $\mathbb{R}^+$ will allow us to swap the "a.s." statement and yield $\mu^\infty \in \mathcal{P}$ almost surely.

Let $t \geq 0$. For all $M \geq 0$, the function

$$f_M : \mu \mapsto \sup_{s\in[0,1]} \int_{\mathbb{R}} (|x| \wedge M)\mu_s(dx)$$

is continuous and bounded on $C([0,+\infty), \mathcal{P}(\mathbb{R}))$. For all $N \geq 1$, the Cauchy-Schwarz inequality as well as Doob's inequality yield:

$$\begin{aligned}
\mathbb{E}\big[f_M(\mu^N)\big] &\leq \frac{1}{N}\sum_{i=1}^{N} \mathbb{E}\left[\sup_{s\in[0,1]} |X_s^{i,N}|\right] \\
&\leq \int_{\mathbb{R}} |x|\; P_0(dx) + \|b\|_\infty + \frac{1}{n}\sum_{i=1}^{n} \left[\mathbb{E}\left(\sup_{s\in[0,1]} \left|\int_0^s \big(\sigma(H*\mu_r^{i,N})(X_r^{i,N})\big)\, dB_r^i\right|^2\right)\right]^{1/2} \\
&\leq C,
\end{aligned}$$

where the constant $C$ does not depend on $M$ nor $N$. As a consequence,

$$\liminf_{M\to+\infty} \mathbb{E}[f_M(\mu^\infty)] \leq C,$$

and using Fatou's lemma, we can write:

$$C \geq \mathbb{E}\left[\liminf_{M\to+\infty} f_M(\mu^\infty)\right] \geq \mathbb{E}\left[\sup_{s\in[0,1]} \liminf_{M\to+\infty} \int_{\mathbb{R}} (|x| \wedge M)\mu^\infty(s)(dx)\right].$$

Finally, by the monotone convergence theorem,

$$\lim_{M\to+\infty} \int_{\mathbb{R}} (|x| \wedge M)\mu^\infty(s)(dx) = \int_{\mathbb{R}} |x|\mu^\infty(s)(dx),$$

so that

$$\mathbb{E}\left[\sup_{s\in[0,1]}\int_{\mathbb{R}}|x|\mu^{\infty}(s)(dx)\right]\leq C,$$

which yields the expected result.

□

We are now only missing one crucial argument, which is the uniqueness of a weak solution to the Cauchy problem (3.9). This will automatically yield the convergence of the sequence of mappings $t\mapsto\mu_t^N$, as we explained earlier.

**Proposition 3.2.5.** *Assume that $A$ is increasing. Then, for all $T>0$ and, in particular, for $T=\infty$, there is at most one weak solution to the Cauchy problem (3.9) in $\mathcal{F}(T)$.*

Before diving into the proof of Proposition (3.2.5), note that adapting this result to the cases of a rank-based model with unbounded drift coefficients was the primary objective of my contribution to this research topic. As we will see, we quickly stumbled upon an obstacle, which was directly due to the unbounded character of our drift, and that we were not able to overcome despite several, interesting ideas.

*Proof.* The general idea behind the proof is to show that uniqueness for the Cauchy problem (3.9) is equivalent to existence for a second, adjoint problem.

Let $T\in(0,\infty]$, and let $F^1$, $F^2\in\mathcal{F}(T)$ be two weak solutions to (3.9). Moreover, for all $0\leq t<T$, define $Q_t:=(0,t)\times\mathbb{R}$. It is immediate that the function $F_t^2-F_t^1$ is integrable over $\mathbb{R}$, and that the function $(s,x)\mapsto F_s^2(x)-F_s^1(x)$ is integrable over $Q_t$. Let us denote $\bar{F}_s(x):=F_s^2(x)-F_s^1(x)$. Combining the weak formulations of the Cauchy problem and using an integrated version of the mean value theorem yields, for any test function $g\in C_c^{\infty}([0,T)\times\mathbb{R})$ :

$$\int_{Q_t}\bar{F}_s(x)\left[\tfrac{1}{2}\widetilde{A}(s,x)\,\partial_x^2 g(s,x)+\widetilde{B}(s,x)\,\partial_x g(s,x)+\partial_s g(s,x)\right]\mathrm{d}s\;\mathrm{d}x \qquad (3.11)$$
$$=\int_{\mathbb{R}}\bar{F}_t(x)\,g(t,x)\;\mathrm{d}x,$$

where we introduced the averaged integral coefficients:

$$\begin{cases}\tilde{A}(s,x) & := \int_0^1 a\big((1-\theta)F_s^1(x)+\theta F_s^2(x)\big)\,d\theta\\ \tilde{B}(s,x) & := \int_0^1 b\big((1-\theta)F_s^1(x)+\theta F_s^2(x)\big)\,d\theta.\end{cases}$$

One can show that, for all $t\in[0,T)$, the equation (3.11) holds true for all $g\in C_b^{1,2}([0,T)\times\mathbb{R})$.

In order to prove $\bar{F}\equiv0$ on $Q_t$, it is enough to show that, for any $f\in C_c^{\infty}((0,t)\times\mathbb{R})$,

$$\int_{[0,t]\times\mathbb{R}}\bar{F}(s,x)\,f(s,x)\,ds\,dx=0, \qquad (3.12)$$

since $C_c^\infty((0,t)\times\mathbb{R})$ is a dense subset of $L^1(Q_t)$. Fix a test function $f \in C_c^\infty((0,t)\times\mathbb{R})$. We wish to solve an adjoint parabolic problem whose solution, which writes:

$$\begin{cases} \frac{1}{2}\tilde{A}\partial_x^2 g + \tilde{B}\partial_x g + \partial_s g = f, & (s,x) \in [0,t)\times\mathbb{R} \\ g(t,x) = 0, & x \in \mathbb{R}, \end{cases} \tag{3.13}$$

where $t$ is such that $\mathrm{Supp}(f) \subset [0,t)\times\mathbb{R}$. Since the coefficients $\tilde{A}$ and $\tilde{B}$ might not be smooth enough in order for the adjoint problem (3.13) to have solutions, we need to regularize and approximate it. Let us introduce two approximation variables $\delta, \eta > 0$, and a partition of the space:

$$G_\delta := \left\{(s,x) \in [0,t]\times\mathbb{R} : |\overline{F}_s(x)| < \delta\right\}, \quad \text{and} \quad F_\delta := ([0,t]\times\mathbb{R}) \setminus G_\delta. \tag{3.14}$$

Since $A$ in increasing and $F^1$, $F^2$ take values in $[0,1]$, there exist positive constants $L(\delta) > 0$ and $K(\delta) > 0$ such that:

$$\widetilde{A}(s,x) \geq L(\delta) \quad \text{on } F_\delta, \tag{3.15}$$

$$|\lambda_\eta^\delta(s,x)| \leq K(\delta) \quad \text{on } [0,t]\times\mathbb{R}, \tag{3.16}$$

where we defined:

$$\lambda_\eta^\delta(s,x) := \begin{cases} 0, & (s,x) \in G_\delta, \\ \dfrac{\widetilde{B}(s,x)}{\sqrt{\frac{1}{2}(\eta + \widetilde{A}(s,x))}}, & (s,x) \in F_\delta. \end{cases} \tag{3.17}$$

In what follows, the values of constants such as $C$ and $K(\delta)$ can change from one line to another. Now, let us introduce a regularization of our coefficients. Let $\xi$ be a $C^\infty$ probability density on $\mathbb{R}^2$ such that

$$\mathrm{Supp}(\xi) \subset [-1,1]\times[-1,1].$$

For all $\epsilon > 0$, let

$$\xi_\epsilon := \epsilon^{-2}\xi(\epsilon^{-1}s, \epsilon^{-1}x),$$

and define

$$\widetilde{A}_\epsilon = \widetilde{A} * \xi_\epsilon, \qquad \lambda_{\eta,\epsilon}^\delta = \lambda_\eta^\delta * \xi_\epsilon.$$

Then $\widetilde{A}_\epsilon$ and $\lambda_{\eta,\epsilon}^\delta$ are $C^\infty$ functions and all of their derivatives are bounded on $[0,t]\times\mathbb{R}$. Moreover, we have:

$$\lim_{\epsilon\to 0} \widetilde{A}_\epsilon(s,x) = \widetilde{A}(s,x) \quad \text{a.e. in } [0,t]\times\mathbb{R}, \tag{3.18}$$

$$\lim_{\epsilon\to 0} \lambda_{\eta,\epsilon}^\delta(s,x) = \lambda_\eta^\delta(s,x) \quad \text{a.e. in } [0,t]\times\mathbb{R}, \tag{3.19}$$

$$\widetilde{A}_\epsilon(s,x) \leq C \quad (s,x) \in [0,t]\times\mathbb{R}, \tag{3.20}$$

$$\left|\lambda_{\eta,\epsilon}^\delta(s,x)\right| \leq K(\delta) \quad (s,x) \in [0,t]\times\mathbb{R}, \tag{3.21}$$

where $C$ refers to a positive constant independent of $\epsilon$, $\delta$ and $\eta$, and $K(\delta)$ refers to a positive constant depending only on $\delta$. We finally define:

$$\widetilde{B}^{\delta}_{\eta,\epsilon}(s,x) = \lambda^{\delta}_{\eta,\epsilon}(s,x)\left[\tfrac{1}{2}\Big(\eta + \widetilde{A}_{\epsilon}(s,x)\Big)\right]^{1/2}. \tag{3.22}$$

Note the fact that:

$$\|\widetilde{B}^{\delta}_{\eta,\epsilon}\|_{\infty} \le K(\delta). \tag{3.23}$$

We are now able to introduce the approximate adjoint problem:

$$\begin{cases} \dfrac{1}{2}\Big(\eta + \widetilde{A}_{\epsilon}\Big)\,\partial^2_x g + \widetilde{B}^{\delta}_{\eta,\epsilon}\,\partial_x g + \partial_s g = f, & (s,x) \in [0,t]\times\mathbb{R}, \\ g(t,x) = 0, & x \in \mathbb{R}. \end{cases} \tag{3.24}$$

At this point, it is easy to check that the approximate adjoint problem has a unique classical solution which is bounded, according to [KS91]. However, the boundedness of the coefficients of the problem is crucial for this step, and our attempts to circumvent it were not successful. The smoothing function $\xi$, though of compact support, does not prevent the unbounded nature of $\widetilde{B}^{\delta}_{\eta,\epsilon}$. Hence, for the rest of the proof, we will only highlight the important ideas behind controlling the solution of the approximate problem, as we were not able to go this far in our own work.

As we mentioned, since the coefficients are bounded, Lipschitz continuous, and the operator is uniformly parabolic, and given that $f$ is continuous and bounded, the approximate adjoint problem (3.24) has a unique classical solution which we will denote by $g^{\delta}_{\eta,\epsilon}$. According to [Fri64, p. 263], $g^{\delta}_{\eta,\epsilon}$ is $C^{\infty}$ on $[0,t]\times\mathbb{R}$. The Feynman-Kac formula yields, for all $(s,x) \in [0,t)\times\mathbb{R}$,

$$g^{\delta}_{\eta,\epsilon}(s,x) = -\mathbb{E}\left[\int_s^t f\big(r,\, Z^{s,x}_r\big)\,\mathrm{d}r\right], \tag{3.25}$$

where, for a given standard Brownian motion $B$, $(Z^{s,x}_r)_{r\in[0,t]}$ is the unique, strong solution of the following SDE: for all $r \in [s,t]$,

$$Z^{s,x}_r = x + \int_s^r \widetilde{B}^{\delta}_{\eta,\epsilon}\big(u,\, Z^{s,x}_u\big)\,\mathrm{d}u + \int_s^r \big(\eta + \widetilde{A}_{\epsilon}(u,\, Z^{s,x}_u)\big)^{1/2}\,\mathrm{d}W_u. \tag{3.26}$$

We now have to introduce a few intermediate results which we will not give proof of. The first one deals with finding suitable upper-bounds on the solution $g^{\delta}_{\eta,\epsilon}$ and its derivatives.

**Lemma 3.2.6.** *For each $\delta, \eta, \epsilon > 0$ there exist constants $C, K(\delta) > 0$ and $\kappa(\varepsilon, \delta, \eta) > 0$ (not necessarily the same from one line to the other), such that:*

$$\sup_{(s,x)\in[0,t]\times\mathbb{R}} |g^{\delta}_{\eta,\epsilon}(s,x)| \leq C, \tag{3.27}$$

$$\sup_{s\in[0,t]} \int_{\mathbb{R}} |g^{\delta}_{\eta,\epsilon}(s,x)|\, dx \leq K(\delta), \tag{3.28}$$

$$\sup_{s\in[0,t]} |\partial_x g^{\delta}_{\eta,\epsilon}(s,x)| \leq \kappa(\varepsilon,\delta,\eta)\, \mathrm{e}^{-x^2/\kappa(\varepsilon,\delta,\eta)}, \tag{3.29}$$

$$\sup_{s\in[0,t]} |\partial^2_x g^{\delta}_{\eta,\epsilon}(s,x)| \leq \kappa(\varepsilon,\delta,\eta)\, \mathrm{e}^{-x^2/\kappa(\varepsilon,\delta,\eta)}. \tag{3.30}$$

Using the fact that $g^{\delta}_{\eta,\epsilon}$ is a solution to (3.24), and that $g^{\delta}_{\eta,\epsilon} \in C^{1,2}_b([0,t]\times\mathbb{R})$ to apply (3.11), we can write:

$$\begin{aligned}
&\int_{[0,+\infty)\times\mathbb{R}} (F^2_s(x) - F^1_s(x)) f(s,x)\, \mathrm{d}s\, \mathrm{d}x \\
&\qquad = \int_{Q_t} (F^2_s - F^1_s) \left[\frac{1}{2}(\eta + \tilde{A}_\epsilon - \tilde{A})\partial^2_x g^{\delta}_{\eta,\epsilon} + (\tilde{B}^{\delta}_{\eta,\epsilon} - \tilde{B})\partial_x g^{\delta}_{\eta,\epsilon}\right] \mathrm{d}s\, \mathrm{d}x.
\end{aligned} \tag{3.31}$$

The goal is now to show that the right-hand side of (3.31) goes to zero when $\delta, \eta, \epsilon \to 0$.

**Lemma 3.2.7.** *There exist constants $K(\delta)$ and $C$ such that for all $\varepsilon, \eta$ as above,*

$$\int_{Q_t} \tfrac{1}{2}(\eta + \widetilde{A}_\varepsilon)\big(\partial^2_x g^{\delta}_{\eta,\epsilon}\big)^2\, dx\, ds \leq \frac{K(\delta)}{\eta} + C, \tag{3.32}$$

$$\int_{Q_t} \big(\partial_x g^{\delta}_{\eta,\epsilon}\big)^2\, dx\, ds \leq \frac{K(\delta)}{\eta} + C. \tag{3.33}$$

**Lemma 3.2.8.** *There exists $C > 0$ independent of $\delta, \eta, \varepsilon$ such that*

$$\sup_{s\in[0,t]} \int_{\mathbb{R}} \big|\partial_x g^{\delta}_{\eta,\epsilon}(s,x)\big|\, dx \leq C. \tag{3.34}$$

These two results are enough to show that the right-hand side of (3.31) goes to zero when $\delta, \eta, \epsilon \to 0$. There is still work to do, and we will not provide more details here. To summarize, we showed that:

$$\int_{[0,T)\times\mathbb{R}} \big(F^2_s(x) - F^1_s(x)\big) f(s,x)\, \mathrm{d}s\, \mathrm{d}x = 0. \tag{3.35}$$

Since $f$ is chosen arbitrarily, we conclude that:

$$F_s^2(x) = F_s^1(x), \quad a.e. \text{ in } Q_t, \tag{3.36}$$

and this for $t \in [0, T)$. In other words, $F^1 = F^2$ in $\mathcal{F}$.

□

## 3.3 The rank-based Ornstein-Uhlenbeck model

Now, we are looking to adapt the work of Jourdain and Reygner in the case of a rank-based model with unbounded drift. More specifically, we intended to show propagation of chaos in a particle system with the drift of an Ornstein-Uhlenbeck process.

In this section, let us consider the following rank-based particle system. For all $i \in \{1, ..., N\}$,

$$\mathrm{d}X_t^{i,N} = -\sum_{k=1}^{N} \left(\lambda_k^N X_t^{i,N} - \eta_k^N\right) \mathbf{1}_{X_t^{i,N} = X_t^{(k)}} \, \mathrm{d}t + \sum_{k=1}^{N} \sigma_k^N \mathbf{1}_{X_t^{i,N} = X_t^{(k)}} \, \mathrm{d}B_t^i, \tag{$OU_N$}$$

where $(B^i)_{1 \leq i \leq N}$ is a collection of independent, standard Brownian motions. In this setting, the drift coefficients of the classic Atlas model are replaced by non-constant coefficients with spatially linear growth. Let $\lambda, \zeta, \sigma$ be real-valued, measurable functions defined on $[0, 1]$ such that:

$$\forall k \leq N, \quad \lambda_k^N = \lambda\left(\frac{k}{N}\right), \ \ \zeta_k^N = \zeta\left(\frac{k}{N}\right), \ \ \sigma_k^N = \sigma\left(\frac{k}{N}\right).$$

Assume further that $\sigma$ is positive on $[0, 1]$. We are looking to adapt some of the methods used in [JR13] in order to prove propagation of chaos in some sense for our particle system.

Our general motivation is to investigate general properties of rank-based models with more complex drift coefficients than the ones considered in previous papers. To the best of our knowledge, there is no pre-existing work on existence and uniqueness of either weak or strong solutions to these generalized Atlas systems, nor on potential chaos propagation properties. We chose to focus on the latter here, assuming that we already have the existence of a weak solution to the rank-based Ornstein-Uhlenbeck equations ($OU_N$), which *a priori* we do not. Prior research by Hélène and her coworker hints at the difficulty of adapting the ideas of [JR13] to a general collection of spatially unbounded drift coefficients $\{b(t, x, \frac{k}{N}); 1 \leq k \leq N\}$. By considering the special case of an Ornstein-Uhlenbeck base (see [OU30]), we hoped to make use of its mean-reverting property to obtain control, in some sense, over the particle system.

### 3.3.1 The limiting SDE and the associated Cauchy problem

Following Jourdain and Reygner's method, we now translate the problem from a probabilistic to a deterministic context. First, let us rewrite the system of SDEs. For all $i \in \{1, ..., N\}$,

$$\mathrm{d}X_t^{i,N} = -\Big(\lambda\left(H * \mu_s^N(X_s^{i,N})\right)X_t^{i,N} - \eta\left(H * \mu_s^N(X_s^{i,N})\right)\Big)\,\mathrm{d}t + \sigma(H * \mu_s^N(X_s^{i,N}))\,\mathrm{d}B_t^i.$$

As we did before, we are looking for a limit to the sequence of empirical measures. Assuming its existence and taking the formal limit in (3.3.1) yields the following limiting SDE:

$$\begin{cases} X_t = X_0 - \int_0^t [\lambda(H * P_s(X_s))X_s - \eta(H * P_s(X_s))]\,\mathrm{d}s + \int_0^t \sigma(H * P_s(X_s))dB_s \\ X_t \sim P_t, \end{cases} \tag{3.37}$$

with initial condition $X_0 \sim P_0$ given and independent from $B$. Once again, we can rewrite the above equation by introducing the CDF of the process $X$ as follows:

$$\begin{cases} X_t = X_0 - \int_0^t [\lambda(F_s(X_s))X_s - \eta(F_s(X_s))]\,\mathrm{d}s + \int_0^t \sigma(F_s(X_s))dB_s, \\ F_t \text{ is the CDF of } X_t. \end{cases} \tag{3.38}$$

We are now looking to find, in a similar approach as before, a deterministic description of the dynamics of the collection of CDFs $\{F_t\}_{t\geq 0}$ of the limiting process. By introducing a test function $f \in C_c^\infty(\mathbb{R})$, we can apply Itô's formula and write, for all $t \geq 0$:

$$\begin{aligned} f(X_t) = f(X_0) + M_t + \frac{1}{2}\int_0^t \sigma^2(F_s(X_s))f''(X_s)\,\mathrm{d}s \\ - \int_0^t [\lambda(F_s(X_s))X_s - \eta(F_s(X_s))]\,f'(X_s)\,\mathrm{d}s, \end{aligned} \tag{3.39}$$

where $(M_t)_{t\geq 0}$ is a martingale with zero expectation. Let us take the expectation in the above equation. Assuming that $P_t$ has density $p_t$ on $\mathbb{R}$, we get:

$$\begin{aligned} \int_{\mathbb{R}} f(x)p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)p_0(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_0^t \sigma^2(F(x))f''(x)p_s(x)\,\mathrm{d}s\,\mathrm{d}x \\ - \int_{\mathbb{R}}\int_0^t [\lambda(F_s(x))x - \eta(F_s(x))]\,f'(x)p_s(x)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{3.40}$$

We can now integrate by parts both terms on the right, using the fact that $f$ has compact support. On the one hand:

$$\begin{aligned} &\int_{\mathbb{R}}\int_0^t [\lambda(F_s(x))x - \eta(F_s(x))]\,f'(x)p_s(x)\,\mathrm{d}s\,\mathrm{d}x \\ &\qquad = -\int_{\mathbb{R}}\int_0^t f(x)\,\partial_x\left[(\lambda(F_s(x))x - \eta(F_s(x)))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

And on the other hand,

$$\int_{\mathbb{R}}\int_0^t \sigma^2(F_s(x))f''(x)p_s(x)\,\mathrm{d}s\,\mathrm{d}x = \int_{\mathbb{R}}\int_0^t f(x)\,\partial_x^2\left[\sigma^2(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x.$$

Combining these terms gives the following equation:

$$\begin{aligned}\int_{\mathbb{R}} f(x)p_t(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)p_0(x)\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}}\int_0^t f(x)\,\partial_x^2\left[\sigma^2(F_s(x))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x \\ + \int_{\mathbb{R}}\int_0^t f(x)\,\partial_x\left[(\lambda(F_s(x))x - \eta(F_s(x)))p_s(x)\right]\,\mathrm{d}s\,\mathrm{d}x,\end{aligned} \tag{3.41}$$

which is once again the weak formulation of a new nonlinear Fokker-Planck equation:

$$\partial_t p_t(x) = \frac{1}{2}\partial_x^2\left[\sigma^2(F_t(x))p_t(x)\right] + \partial_x\left[(\lambda(F_t(x))x - \eta(F_t(x)))p_t(x)\right]. \tag{3.42}$$

This new equation is more complex to approach than the one in Section 3.2. The Cauchy problem we will be studying in this case can be retrieved by integrating over space, knowing that for all $t \geq 0,\ \partial_x F_t(.) = p_t(.)$:

$$\begin{cases}\partial_t F_t(x) = \frac{1}{2}\partial_x^2\left[A(F_t(x))\right] + \partial_x\left[B_L(x, F_t(x))\right] - \Lambda(F_t(x)) \\ F_0(x) = H * P_0(x),\end{cases} \tag{3.43}$$

where we defined:

$$\begin{cases}A(u) := \int_0^u \sigma^2(v)dv \\ \Lambda(u) := \int_0^u \lambda(v)dv \\ H(u) := \int_0^u \eta(v)dv \\ B_L(x,u) := \Lambda(u)x - H(u).\end{cases}$$

Introducing a test function $g \in C_c^\infty([0,t]\times\mathbb{R})$ allows us to obtain a weak formulation of the Cauchy problem as before. After integrating by parts over time on the left-hand side, and over space on the right-hand side:

$$\begin{aligned}\int_{\mathbb{R}} g(t,x)F_t(x)\,\mathrm{d}x - \int_{\mathbb{R}} g(0,x)H * P_0(x)\,\mathrm{d}x \\ = \int_{\mathbb{R}}\int_0^t \Big[F_s(x)\,\partial_s g(s,x) + \frac{1}{2}A(F_s(x))\,\partial_x^2 g(s,x) \\ - B_L(x, F_s(x))\,\partial_x g(s,x) - \Lambda(F_s(x))\,g(s,x)\Big]\,\mathrm{d}s\,\mathrm{d}x\end{aligned} \tag{3.44}$$

As mentioned earlier, studying this Cauchy problem and showing uniqueness of a weak solution was the main focus point of my research during this internship. We were quickly convinced that showing the tightness of our sequence of distributions and characterizing the limit of a converging subsequence would not pose major difficulties. We thus concentrated our effort on studying the Cauchy problem directly with the goal of circumventing the unboundedness issue. Hélène and her collaborator stumbled upon the same obstacle a few years ago while tackling the problem for a general, unbounded set of drift coefficients of the form:

$$b_k^N(t,x) = b\bigg(\frac{k}{N}, t, x\bigg).$$

In the hopes that the properties of Ornstein-Uhlenbeck processes might help us overcome this difficulty, we tried to directly adapt the proof of [JR13], unsuccessfully. Following the idea of Jourdain and Reygner, let us introduce two weak solutions $F^1$ and $F^2$ to the Cauchy problem. By substracting the weak formulations for $F^1$ and $F^2$, we can write, for any test function $g \in C_c^\infty([0,T) \times \mathbb{R})$:

$$\int_{\mathbb{R}} g(t,.)\bar{F}_t = \iint_{Q_t} \bar{F}_s(x)\bigg[\frac{1}{2}\partial_x^2\big[A(F_s^2(x)) - A(F_s^1(x))\big] - \partial_x\big[B_L(x,F_s^2(x)) - B_L(x,F_s^1(x))\big] - \big[\Lambda(F_s^2(x)) - \Lambda(F_s^1(x))\big] + \partial_s g\bigg]\,\mathrm{d}s\,\mathrm{d}x, \quad (3.45)$$

where we recall that $\bar{F}_t := F_t^2 - F_t^1$. Let us also denote $F_s^\theta(x) := \theta F_s^2(x) + (1-\theta)F_s^1(x)$. Then, we can introduce the following quantities:

$$\tilde{A}(s,x) := \int_0^1 \sigma^2\big(F_s^\theta(x)\big)\,\mathrm{d}\theta; \quad (3.46)$$

$$\tilde{B}_L(s,x) := \int_0^1 \partial_u B_L(x, F_s^\theta(x))\,\mathrm{d}\theta; \quad (3.47)$$

$$\tilde{\Lambda}(s,x) := \int_0^1 \lambda(F_s^\theta(x))\,\mathrm{d}\theta. \quad (3.48)$$

Reassembling these elements into the substracted weak formulations then yields the adjoint problem: finding $g : [0,t] \times \mathbb{R} \to \mathbb{R}$ such that

$$\begin{cases} \partial_s g + \frac{1}{2}\tilde{A}\partial_x^2 g + \tilde{B}_L\partial_x g + \tilde{\Lambda}g = f \in C_c^\infty([0,t) \times \mathbb{R}) \\ g(t,.) = 0 \end{cases} \quad (3.49)$$

### 3.3.2 Attempting to trap the coefficients inside a box

From there, we already have an issue. Although Jourdain and Reygner introduce an approximation of their adjoint problem, which we could also do, the problem remains that we do not know how to prove that such a problem has a (unique) solution. Once again, the issue resides in the unboundedness of $\tilde{B}_L$. One of the ideas we investigated was to try and prove that, in practice, $x$ is necessarily confined in a bounded subset of $\mathbb{R}$. We can show such a result in the case of the standard Atlas model:

**Lemma 3.3.1.** *Under the same assumptions as Theorem 3.2.2, there exists a nondecreasing function $C : t \mapsto C(t) > 0$ such that:*

$$\forall x \leq -C(t), \quad \frac{1}{2}F_0(2x - C(t)) \leq F_t(x) \leq F_0\Big(\frac{x}{2} + C(t)\Big) + \exp\left(-\frac{x^2}{C(t)}\right); \tag{3.50}$$

$$\forall x \geq C(t), \quad \frac{1}{2}[1 - F_0(2x + C(t))] \leq 1 - F_t(x) \leq 1 - F_0\Big(\frac{x}{2} - C(t)\Big) + \exp\left(-\frac{x^2}{C(t)}\right). \tag{3.51}$$

Recall the definition of the coefficients of the approximate adjoint problem:

$$\widetilde{B}^{\delta}_{\eta,\epsilon}(s,x) = \lambda^{\delta}_{\eta,\epsilon}(s,x)\left[\tfrac{1}{2}\Big(\eta + \widetilde{A}_{\epsilon}(s,x)\Big)\right]^{1/2}, \tag{3.52}$$

where we also defined

$$\lambda^{\delta}_{\eta}(s,x) := \begin{cases} 0, & (s,x) \in G_{\delta}, \\ \dfrac{\widetilde{B}(s,x)}{\sqrt{\frac{1}{2}(\eta + \widetilde{A}(s,x))}}, & (s,x) \in F_{\delta}. \end{cases} \tag{3.53}$$

From these definitions, and recalling that $F_{\delta} := \{(s,x) \in [0,t] \times \mathbb{R} : |F^2_s(x) - F^1_s(x)| \geq \delta\}$, we deduce that, for an appropriate choice of $\delta$, we necessarily confined our variables in a bounded box, *i.e.*

$$F_{\delta} \subset [0,t] \times [-C(t), C(t)]. \tag{3.54}$$

Which would directly imply the boundedness of the regularized coefficients $\widetilde{B}^{\delta}_{\eta,\epsilon}(s,x)$, since their support is in $F_{\delta}$ by construction. The proof of Lemma 3.3.1, although quite long, is fairly straightforward and can be adapted from [JR13, Section 3.2]. Let us justify that we can indeed find $M > 0$ such that, for all $|x| \geq M$,

$$\left|F^2_s(x) - F^1_s(x)\right| \leq \left|F^2_s(x) - 1\right| + \left|1 - F^1_s(x)\right| < \delta. \tag{3.55}$$

On the one hand, it is immediate that there exists $M^+ > 0$ such that for all $x \geq M^+$:

$$1 - F_0\Big(\frac{x}{2} - C(t)\Big) < \frac{\delta}{4}, \tag{3.56}$$

$$\exp\left(-\frac{x^2}{C(t)}\right) < \frac{\delta}{4}. \tag{3.57}$$

Hence for all $x \geq M^+$, $1 - F^1_s(x) < \frac{\delta}{2}$ (resp. $F^2_s$). On the other hand, there exists $M^- > 0$ such that, for all $x \leq -M^-$:

$$F_0\Big(\frac{x}{2} + C(t)\Big) < \frac{\delta}{4}, \tag{3.58}$$

$$\exp\left(-\frac{x^2}{C(t)}\right) < \frac{\delta}{4}. \tag{3.59}$$

Hence, for all $x \leq M^-$, $F^1_s(x) < \frac{\delta}{2}$ (resp. $F^2_s$). Let $M(\delta) := \max(M^+, M^-)$. Then, for all $|x| \geq M(\delta)$:

- $*$ if $x \geq M(\delta)$, $\left|\bar{F}_s(x)\right| \leq |F_s^2(x) - 1| + |F_s^1(x) - 1| \leq \delta$;
- $*$ if $x \leq -M(\delta)$, $\left|\bar{F}_s(x)\right| \leq |F_s^2(x)| + |F_s^1(x)| \leq \delta$.

To conclude, for all $x \in \mathbb{R}$, for all $s \leq t$, $|F_s^2(x) - F_s^1(x)| > \delta \Rightarrow |x| < M$. After making the bound uniform in $s \in [0,t]$, we finally get:

$$F_\delta \subset [0,t] \times [-M(\delta), M(\delta)]. \tag{3.60}$$

Although we are able to prove this result in the case of a standard Atlas model, we were unable to prove it in the context of a rank-based Ornstein-Uhlenbeck process. Nonetheless, the idea is interesting, and would deserve further investigation by working directly on the tail of the cumulative distribution function of our process. This is a first perspective for future developments.

### 3.3.3 Comparison theorems and tail estimates

Another idea, which still has to do with the tail of the CDF of the rank-based Ornstein-Uhlenbeck process, was to look into the comparison theorems referenced by Ikeda&Watanabe (see [IW89, Chapter VI]). The first such theorem is interesting in our case. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a reference filtration $(\mathcal{F}_t)_{t\geq 0}$. First, suppose we have the following objects:

(i) A strictly increasing function $\rho : [0,\infty) \to [0,\infty)$ such that $\rho(0) = 0$ and

$$\int_{0+} \rho(\xi)^{-2}\, d\xi = \infty; \tag{3.61}$$

(ii) A continuous function $\sigma : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ such that, for all $t \geq 0$ and all $x, y \in \mathbb{R}$,

$$\big|\sigma(t,x) - \sigma(t,y)\big| \leq \rho\big(|x-y|\big); \tag{3.62}$$

(iii) Two continuous functions $b_1, b_2 : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ satisfying

$$b_1(t,x) < b_2(t,x) \quad \text{for all } t \geq 0,\ x \in \mathbb{R}. \tag{3.63}$$

**Theorem 3.3.2.** *Let $X^1$ and $X^2$ be two real-valued, continuous and $\mathcal{F}_t$-adapted processes, and let $B$ be a one-dimensional $\mathcal{F}_t$-Brownian motion. Finally let $\beta^1$ and $\beta^2$ be two other $\mathcal{F}_t$-adapted and measurable processes. Assume that these processes satisfy, almost surely, the following conditions:*

*(i)* $X_t^i - X_0^i = \int_0^t \sigma\big(s, X_s^i\big)\, \mathrm{d}B(s) + \int_0^t \beta_s^i\, \mathrm{d}s, \quad i = 1,2;$

*(ii)* $X_0^1 \leq X_0^2$;

*(iii)* $\beta_t^1 \leq b_1\big(t, X_t^1\big)$ *for every* $t \geq 0$;

*(iv)* $\beta_t^2 \geq b_2\big(t, X_t^2\big)$ *for every* $t \geq 0$.

*Then, with probability one, we have*

$$X_t^1 \leq X_t^2 \quad \text{for all } t \geq 0. \tag{3.64}$$

*Moreover, if pathwise uniqueness of solutions holds for at least one of the stochastic differential equations*

$$dX(t) = \sigma\big(t, X(t)\big)\, dB(t) + b_i\big(t, X(t)\big)\, dt, \qquad i = 1, 2, \tag{3.65}$$

*then the same conclusion (3.64) remains valid under the weakened assumption*

$$b_1(t, x) \leq b_2(t, x) \quad \text{for all } t \geq 0,\ x \in \mathbb{R}. \tag{3.66}$$

Our goal is to apply this theorem in order to study the following SDE:

$$\begin{cases} X_t = X_0 - \int_0^t \left[\lambda(H * P_s(X_s))X_s - \eta(H * P_s(X_s))\right] \mathrm{d}s + \int_0^t \sigma(H * P_s(X_s)) dB_s \\ X_t \sim P_t, \end{cases} \tag{3.67}$$

in order to compare the rank-based Ornstein-Uhlenbeck with classic Ornstein-Uhlenbeck processes, for which we can characterize the distribution precisely. We can introduce the following modified drift coefficients:

$$\begin{cases} b^-(x) := -\lambda^+ x + \eta^- \\ b^+(x) := -\lambda^- x + \eta^+, \end{cases} \tag{3.68}$$

where we defined $\lambda^+ := \sup\limits_{x\in[0,1]} \lambda(x)$, and $\lambda^- := \inf\limits_{x\in[0,1]} \lambda(x)$ (resp. $\eta^+$ and $\eta^-$). The resulting "framing" processes write:

$$X_t^+ = X_0 + \frac{1}{2}\int_0^t \sigma(F_s(X_s^+))\, \mathrm{d}B_s + \int_0^t b^+(X_s^+)\, \mathrm{d}s \tag{3.69}$$

$$X_t^- = X_0 + \frac{1}{2}\int_0^t \sigma(F_s(X_s^-))\, \mathrm{d}B_s + \int_0^t b^-(X_s^-)\, \mathrm{d}s \tag{3.70}$$

There is still work to do in order to interpret the application of Theorem 3.3.2 and obtain proper control over the distribution of our rank-based Ornstein-Uhlenbeck process. This is our main perspective for developments on this topic, which opens a window for future collaboration with Hélène and Dante.

# References


[Bil99] Patrick Billingsley. *Convergence of Probability Measures.* Wiley, New York, 2nd edition, 1999.

[BP87] Richard F. Bass and Étienne Pardoux. Uniqueness for diffusions with piecewise constant coefficients. *Probab. Theory Related Fields*, 76(4):557–572, 1987.

[CD22] Louis-Pierre Chaintron and Antoine Diez. Propagation of chaos: a review of models, methods and applications. I. models and methods. *Kinetic and Related Models*, 15(6), 2022.

[Che01] A. S. Cherny. On the uniqueness in law and the pathwise uniqueness for stochastic differential equations. *Theory of Probability and its Applications*, 46:483–497, 2001.

[CM44] Robert H. Cameron and William T. Martin. Transformation of wiener integrals under translations. *Annals of Mathematics*, 45:386–396, 1944.

[Fer02] E. Robert Fernholz. *Stochastic Portfolio Theory.* Springer, New York, 2002.

[FIKP12] E. Robert Fernholz, Tomoyuki Ichiba, Ioannis Karatzas, and Vilmos Prokaj. Planar diffusions with rank-based characteristics: Transition probabilities, time reversal, maximality and perturbed tanaka equations. *Stochastic Process. Appl.*, 122(4):1601–1635, 2012.

[Fri64] Avner Friedman. *Partial Differential Equations of Parabolic Type.* Prentice-Hall, Englewood Cliffs, NJ, 1964.

[IKS13] Tomoyuki Ichiba, Ioannis Karatzas, and Mykhaylo Shkolnikov. Strong solutions of stochastic equations with rank-based coefficients. *Probab. Theory Related Fields*, 156(1-2):229–248, 2013.

[IW89] Nobuyuki Ikeda and Shinzo Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24 of *North-Holland Mathematical Library.* North-Holland, 2nd edition, 1989.

[Jou00] Benjamin Jourdain. Probabilistic approximation for a porous medium equation. *Stochastic Processes and their Applications*, 89(1):81–99, 2000.

[JR13] Benjamin Jourdain and Julien Reygner. Propogation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation. *Stoch. Partial Differ. Equ. Anal. Comput.*, 1(3):455–506, 2013.

[Kac56] Mark Kac. Foundations of kinetic theory. In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability*, volume 3, pages 171–197, Berkeley and Los Angeles, California, 1956. University of California Press.

[KR62] Mark G. Krein and Mark A. Rutman. Linear operators leaving invariant a cone in a Banach space. *American Mathematical Society. Selected Translations. Series 1*, 10:199–325, 1962.

[KS84] Ioannis Karatzas and Steven E. Shreve. Trivariate density of brownian motion, its local and occupation times, with applications to stochastic control. *The Annals of Probability*, 12(3):819–828, 1984.

[KS91] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer, New York, 2nd edition, 1991.

[OU30] Leonard S. Ornstein and George E. Uhlenbeck. On the theory of brownian motion. *Physical Review*, 36(5):823–841, 1930.

[SV79] Daniel W. Stroock and Srinivasa Varadhan. *Multidimensional Diffusion Processes*. Springer, Berlin, Heidelberg, New York, 1979.

[Szn91] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d'Été de Probabilités de Saint-Flour XIX – 1989*, volume 1464 of *Lecture Notes in Mathematics*, pages 165–251. Springer, Berlin, 1991.