Internship Report

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Long-time behavior of animal movement models

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1 Introduction

Recent research focusing on the theoretical understanding of the collective behavior exhibited by specific living systems has received considerable attention, as it allows group dynamics to be studied on the basis of a set of simple rules that apply to each agent. Early efforts to model alignment, such as the model proposed by Viscek et al. in [VCBJ+95], have paved the way for subsequent advancements, including the Cucker-Smale model introduced by Cucker and Smale in [CS07], as well as various later variants. These models integrate three primary effects: long-distance attraction, short-distance repulsion, and alignment. These effects supported by empirical studies [Gia08], are called first principles of swarming. Notably, [PKH10] extends such models by incorporating a collision avoidance term. This framework has been subject to numerous adaptations to suit specific contexts, as in [CK03] to model the movements of bird. Unlike animal groups such as small fish or birds, there are groups such as cattle, for example, where social interactions between agents are no longer symmetrical. In particular, Motsch and Tadmor [MT11] (2011) proposed weighing the influence of one agent on another based on the total influence exerted on the latter. In this context, we consider here a variant of the Cucker-Smale's model on a weighted directed graph to take into account social interactions.

Notations In this report, we will use the following notations:

- $\mathbb{M}_N(\mathbb{R})$ is the space of $N \times N$ matrices with real coefficients.
- $\langle \cdot, \cdot \rangle$ is the euclidean scalar product on $\mathbb{R}^d$.
- Id_N is identity matrix of size N.
- $||\cdot||_2$ is the Euclidean norm on $\mathbb{R}^d$ and $||\cdot||_\infty$ is defined for $x \in \mathbb{R}^d$, as $||x||_\infty := \sup\limits_{1\leq i\leq d} |x_i|$. We also define the infinite norm for an application $f : \mathbb{R}^d \to \mathbb{R}$ as $||f||_\infty := \sup\limits_{x\in\mathbb{R}^d} |f(x)|$.
- For $x, y \in \mathbb{R}$, we use the notations $x \vee y := \max\{x, y\}$ and $x \wedge y := \min\{x, y\}$.

2 Cucker-Smale model on a weighted directed graph

In this section, we present an extension of the Cucker-Smale model that was introduced by Cotil in [Cot23] that allow to take into account social interactions that exist within a group of individuals. Let $N \in \mathbb{N}$ be the number of individuals, $x_i(t) := (x_i^1(t), \dots, x_i^d(t)) \in \mathbb{R}^d$ $v_i(t) := (v_i^1(t), \dots, v_i^d(t)) \in \mathbb{R}^d$ be respectively the position and the velocity of agent $i \in \{1, \cdots, N\}$ at time t. We note $(x, v) := (x_i, v_i)_{1\leq i\leq N}$ the couple solution of the following system of Ordinary Differential Equations (ODEs):

$$\begin{cases} \frac{\mathrm{d}x_i}{\mathrm{d}t}(t) &= v_i(t) \\ \frac{\mathrm{d}v_i}{\mathrm{d}t}(t) &= \sum\limits_{j=1}^{N} A_{i,j}\psi(||x_i(t) - x_j(t)||_2)(v_j(t) - v_i(t)), \end{cases} \tag{1}$$

where A is the adjacency matrix of the interaction graph (i.e. $A_{i,j}$ represents the strength of the influence agent j has on agent i or, equivalently, the propensity of agent i to follow agent j). The function ψ called the *communication function*, is supposed to be decreasing (the further two individuals are, the less they communicate), non-negative, locally lipschitz (to ensure the existence of global solutions to (1)) and such that $||\psi||_\infty \leq 1$. A classical form of this function is $\psi : r \mapsto (1 + r^2)^{-\beta/2}$ with $\beta \in \mathbb{R}_+$. We note Q_t the matrix defined by

$$Q_t(i, j) := \begin{cases} A_{i,j}\psi(||x_i(t) - x_j(t)||_2) \text{ if } i \neq j \\ -\sum\limits_{i\neq j} A_{i,j}\psi(||x_i(t) - x_j(t)||_2) \text{ if } i = j. \end{cases} \tag{2}$$

Note that Q is an intensity matrix. Under the above assumptions, the ODE (1) is well posed.

Theorem 2.1 ([Cot23] Theorem 1.1)**.** *The Cucker-Smale model on a weighted directed graph (called the CS model in the following) defined by Equations (1) has a unique solution on* $\mathbb{R}^+$.

One of the most important questions concerning such a model is its long-term behavior, in particular whether all individuals end up moving in the same directions at the same velocity or not. If this is the case, we say that there is flocking, which can be defined as follows.

Definition 2.2. *We say that a solution $(x(t), v(t))$ of Equation (1) flocks if:*

$$\begin{cases} \sup_{t\geq 0} X(t) < \infty \\ V(t) \underset{t\to+\infty}{\to} 0, \end{cases} \tag{3}$$

where $X(t) := \sup_{i,j} ||x_i(t) - x_j(t)||_2$ *and* $V(t) := \sup_{i,j} ||v_i(t) - v_j(t)||_2$.

Our goal is to determine a necessary and sufficient conditions for observing this global alignment or not. In the case where ψ is assumed to be positive on $\mathbb{R}_+$, [MT11] shows that a sufficient condition is

$$V(0) < \int_{X_0}^{+\infty} \psi(r)dr.$$

This condition is critical in the sense that if the integral of ψ diverges (we thus say that ψ has a heavy tail), then flocking will occur regardless of the initial conditions. On the other hand, if the latter converges, then we can always find initial conditions such that flocking does not occur. In this report, we work under the following assumption:

$$\int_0^{+\infty} \psi(r)dr < +\infty. \tag{4}$$

Remark 2.3. *In the following, we will always assume that the graph associated with the adjacency matrix has a single recurrent class. If this is not the case, we can always find initial conditions such that flocking does not occur.*

3 Flocking and non-flocking conditions of the Cucker-Smale model

3.1 Two agent case

Let (x, v) be a solution to Equation (1). Intuitively, flocking depends on a trade-off between the initial conditions (x_0, v_0), the communication function ψ and the interaction matrix A. For the sake of simplicity we first focus our attention on the case of two agents. We consider (x_1, x_2, v_1, v_2) satisfying the following system:

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t}(t) &= v_1(t) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t}(t) &= v_2(t) \\ \frac{\mathrm{d}v_1}{\mathrm{d}t}(t) &= A_1\psi(||x_1(t) - x_2(t)||_2)(v_2(t) - v_1(t)) \\ \frac{\mathrm{d}v_2}{\mathrm{d}t}(t) &= A_2\psi(||x_1(t) - x_2(t)||_2)(v_1(t) - v_2(t)), \end{cases} \tag{5}$$

where $A_1 \geq 0, A_2 \geq 0$ and $A := A_1 + A_2 > 0$. Recall that A_1 (resp. A_2) is the influence of agent 1 on agent 2 (resp. of agent 2 on agent 1). The following theorem gives necessary and sufficient conditions of flocking.

Theorem 3.1. *If (x_1, v_1, x_2, v_2) satisfies (5), then the system flocks if and only if*

$$V(0) < A\int_{\langle \overline{v}_0, x_0\rangle}^{+\infty} \psi_{Y_0}(s)ds,$$

where $x_0 = x_1(0) - x_2(0), v_0 = v_1(0) - v_2(0), \overline{v}_0 = \frac{v_0}{||v_0||_2}, Y_0 = \sqrt{||x_0||_2^2 - \langle \overline{v}_0, x_0\rangle^2}$ *and* $\psi_{Y_0}(s) := \psi\left(\sqrt{Y_0^2 + s^2})\right)$.

Proof. The proof can be found in the Master internship report of Cotil, [Theorem 2.3 [Cot21]]. The idea is to introduce $x = x_1 - x_2$ and $v = v_1 - v_2$, which verify the system of dissipative differential equations (SDDE) below:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t}(t) &= v(t) \\ \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= -A\psi(||x(t)||_2)v(t). \end{cases} \tag{6}$$

We do not detail the full proof in this section as it is very similar to the proof of Theorem 4.2. □

Note that this theorem provides necessary and sufficient flocking conditions in function of $V(0)$, the interaction matrix and the communication function. In the general case (more than 2 agents) whether there is flocking or not can not depend only on the initial diameter of the velocities, as it does not take into account the angles between all the $v_i(0)$. In the next section, we will give sufficient conditions for flocking and non flocking.

3.2 Tools to study the flocking in the general case

In the following sections, (x, v) will denote a solution to CS Equation (1), and (X, V) their associated diameters (defined in Definition 2.2). In [Cot23], Cotil used a probabilistic interpretation to find flocking conditions, showing the link between the convergence of the contraction rate (Dobroshin's ergodicity coefficient) of some Markov chain and flocking.

3.2.1 Dobroshin's ergodicity coefficient

We begin by reviewing the framework and a few results before extending some of them. Let the function $P : (s,t) \mapsto P_{s,t} \in \mathbb{M}_N(\mathbb{R})$ be the unique solution of $\forall 0 \le s \le t$,

$$\begin{cases} P_{t,t} &= I_N \\ \partial_t P_{s,t} &= Q_t P_{s,t} \\ \partial_s P_{s,t} &= -Q_s P_{s,t}, \end{cases} \tag{7}$$

where Q is the intensity matrix given by Equation (2). This function satisfies the "semi-group property", i.e. if v is a solution of (1), then $\forall s \le u \le t$,

$$v(t) = P_{s,t} v(s) \text{ and } P_{s,t} = P_{s,u} P_{u,t}. \tag{8}$$

Remark 3.2. *For all* $0 \le s \le t$,

$$P_{s,t} = P_{0,t-s}.$$

For $z \in \mathbb{R}^d$ and $y \in \mathbb{R}^{Nd}$, $z \cdot y \in \mathbb{R}^{Nd}$ is defined by $(z \cdot y)_i^p = z_p y_i^p$, $1 \le i \le N$ and $1 \le p \le d$. We introduce the linear space:

$$H := \{(z, \dots, z)^T \in \mathbb{R}^{Nd} : z \in \mathbb{R}^d\} = \{z \cdot \mathbb{1} : z \in \mathbb{R}^d\},$$

where $\mathbb{1} := (\underbrace{1, \dots, 1}_{\text{Nd times}})$. For $y \in \mathbb{R}^{Nd}$, we also define $||y||_H := 2 \inf_{h \in H} ||y - h||_\infty$. Let us recall some known results on the semi-norm $|| \cdot ||_H$.

Lemma 3.3 (Lemma 2.1 [Cot24])**.** *$\forall y \in \mathbb{R}^{Nd}$, we have:* $||y||_H = \max_{i,j} ||y_i - y_j||_\infty$.

Theorem 3.4. *[[GQ15] Theorem 6.2] Let us consider the subordinate norm related to $|| \cdot ||_H$ defined for all stochastic matrices $M \in \mathbb{M}_N(\mathbb{R})$ by $||M||_H^{op} := \inf\{\lambda \in \mathbb{R}_+ : \forall y \in \mathbb{R}^N, ||My||_H \le \lambda ||y||_H\}$. We have*

$$||M||_H^{op} = 1 - \mu(M),$$

where $\mu(M) := \min_{i \neq j} \sum_{k=1}^N M_{i,k} \wedge M_{j,k}$ is called the Dobroshin's ergodicity coefficient of the matrix M.

Remark 3.5. *Since $|| \cdot ||_H$ is an operator norm, we easily observe that $|| \cdot ||_H^{op}$ is sub-multiplicative: for stochastic matrices $Q, R \in \mathbb{M}_N(\mathbb{R})$,*

$$||QR||_H^{op} \le ||Q||_H^{op} ||R||_H^{op}.$$

Moreover $||\cdot||_H$ characterizes the contraction of the velocity diameter. Indeed, for (x,v) solution of ODE (1), for all $0 \le s \le t$,

$$||v(t)||_H \le ||P(s,t)||_H^{op}||v(s)||_H.$$

The ergodicity coefficient for a stochastic matrix P measures the distance between a solution to CS equation (1) and a flocking situation as a function of time.

Proposition 3.6. *We have for $0 \le s \le t$,*

$$V(t) \le (1-\mu(P_{s,t}))V(s),$$

where μ is defined in Theorem 3.4 and V is defined in Definition 2.2.

Proof. This is a consequence of the semi-group property (8), of the characterization of the semi-norm $||\cdot||_H$ given in Lemma 3.3, and of the relation between the associated operator norm and the Dobroshin's coefficient given in Theorem 3.4. □

Proposition 3.7. *The function $t \mapsto V(t) := ||v(t)||_H$ is decreasing on $\mathbb{R}_+$.*

Proof. By [She08, Theorem 4.2 (i)], let $0 \le s \le t$ and C a closed convex set of $\mathbb{R}^d$, then

$$\forall i \in \{1,\dots,\mathbb{N}\}, v_i(s) \in C \Rightarrow \forall i \in \{1,\dots,\mathbb{N}\}, v_i(t) \in C.$$

Considering C as the convex envelope of $\{v_i(s)\}_{i\in\{1,\dots,\mathbb{N}\}}$, we deduce that $V(t) \le V(s)$. □

Lemma 3.8. *If (x,v) be a solution of the Cucker-Smale model, then*

$$\forall t \ge 0, V(t) = 0 \Leftrightarrow \exists t \ge 0, V(t) = 0.$$

Proof. *(i)* The first implication is trivial.
(ii) Suppose there exists $t_0 \ge 0$ such that $V(t_0) = 0$.
Then, $\forall t \ge t_0, V(t) = 0$, because V is a positive decreasing function of time by Proposition 3.7. The functions $(\widetilde{x},\widetilde{v})$ defined $\forall t \in [0,t_0]$ as $\widetilde{x}(t) := x(t_0-t), \widetilde{v}(t) := v(t_0-t)$ satisfy the following Equation:

$$\begin{cases} \frac{\mathrm{d}\widetilde{x}_i}{\mathrm{d}t}(t) &= -\widetilde{v}_i(t) \\ \frac{\mathrm{d}\widetilde{v}_i}{\mathrm{d}t}(t) &= -\sum_{j=1}^{N} A_{i,j}\psi(||\widetilde{x}_i(t)-\widetilde{x}_j(t)||_2)(\widetilde{v}_j(t)-\widetilde{v}_i(t)). \end{cases}$$

Using an argument similar to the one presented in Proposition 3.7, we find that $\widetilde{V} : t \to V(t_0-t)$ is decreasing and $\widetilde{V}(0) = 0$ allows us to conclude. □

3.2.2 Link between SDDI and ergodicity coefficient

A classical way to show that a solution is flocking is to show that the diameter in position and velocity satisfy a System of Dissipative Differential Inequalities (SDDI).

Definition 3.9. *Let f and g be two functions from $\mathbb{R}_+$ to $\mathbb{R}_+$. We say that the couple (f,g) satisfies a SDDI if f and g are continuous and piecewise continuously differentiable and if there exists $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $t \ge 0$ where f and g are differentiable:*

$$\begin{cases} \frac{df}{dt}(t) & \le f(t) \\ \frac{dg}{dt}(t) & \le -\phi(f(t))g(t). \end{cases} \tag{9}$$

Proposition 3.10. *[Extension of Grönwall's Lemma] Let f and g be two continuous and piecewise continuously differentiable functions which satisfy Equations (9) for a given positive and decreasing function ϕ. Let us assume that*

$$g(0) < \int_{f(0)}^{+\infty} \phi(r)\mathrm{d}r.$$

Then, setting $f_M > 0$ such that

$$g(0) = \int_{X_0}^{f_M} \phi(r)\mathrm{d}r,$$

we have $f(t) \le f_M$ and $g(t) \le g(0)\mathrm{e}^{-\phi(f_M)t}$ for any $t \ge 0$.

Remark 3.11. *To show that there is flocking, it is enough to find a couple (f, g) that satisfy a SDDI, and such that $X \leq f$ and $V \leq g$. From Proposition 3.10, we easily deduce that (X, V) flocks in the sense of Definition 2.2.*

Two other methods for finding flocking conditions presented in [Cot23] use the two propositions below. These propositions use the characterization of the Dobroshin's ergodicity coefficient as the contraction rate of the velocity diameter (see Proposition 3.6).

Proposition 3.12 (Proposition 4.3 in [Cot23])**.** *Let us suppose that there exists a function $C : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that*

$$\begin{cases} \forall t \geq 0, r \mapsto C(t,r) \text{ is increasing,} \\ \forall t \geq 0, 1 - \mu(P_{0,t}) \leq C\left(t, \sup_{s \leq t} X(s)\right), \\ \forall r \geq X_0, C(t,r) \underset{t \to +\infty}{\to} 0, \end{cases} \tag{10}$$

where the Dobroshin's ergodicity coefficient μ is defined in Theorem 3.4 and X is defined in Definition 2.2. Then, finding $r_0 \geq X_0$ such that

$$r_0 - V(0) \geq V(0) \int_0^{+\infty} C(s, r_0) \mathrm{d}s$$

leads to flocking.

Proposition 3.13 (Proposition 4.4 in [Cot23])**.** *Let us suppose that there exists $C : \mathbb{R}_+^2 \to \mathbb{R}_+$ such that*

$$\begin{cases} \forall t \geq 0, r \mapsto C(t,r) \text{ is decreasing,} \\ \forall s \leq t \geq 0, \mu(P_{s,t}) \geq C\left(t - s, \sup_{u \leq t} X(u)\right), \end{cases} \tag{11}$$

Then, finding $r_0 \geq X_0$ and $t_0 \geq 0$ such that

$$r_0 - V(0) \geq V(0) \frac{t_0}{C(t_0, r_0)}$$

leads to flocking.

Proposition 3.14. *The first two assumptions in Proposition 3.12 and Proposition 3.13 are equivalent. Assuming that for all $r \geq X_0$, $C(t,r) \underset{t \to +\infty}{\to} 0$ in Proposition 3.13 only serves to loosen the constraints on the initial conditions while guaranteeing flocking.*

Proof. Let's consider a function C satisfying the hypothesis of Proposition 3.12. We define $\widetilde{C}(t,r) := 1 - C(t,r)$. Then $\widetilde{C}$ satisfies the condition of Proposition 3.13:

- $\forall t \geq 0$, $r \mapsto \widetilde{C}(t,r) = 1 - C(t,r)$ is decreasing.
- Let $s \leq t$, we have $P_{s,t} = P_{0,t-s}$, thus

  $$1 - \mu(P_{0,t-s}) = 1 - \mu(P_{s,t}) \leq C(t-s, \sup_{u \leq t-s} X(u)) \leq C(t-s, \sup_{u \leq t} X(u))$$

  since C is increasing. Consequently, $\mu(P_{s,t}) \geq \widetilde{C}(t-s, \sup_{u \leq t} X(u))$.

The equivalence holds when going backward in the inequalities. □

Proposition 3.15. *Let (x, v) be the solution of Equation (1) and let (X, V) be defined as in Definition 2.2. Assume that the hypothesis of Proposition 3.13 holds and suppose that for every $r \geq 0$,*

$$\lim_{\epsilon \to 0^+} \frac{1 - C(\epsilon, r)}{\epsilon} < \infty \text{ and } \forall r \geq 0, t \geq 0, \ C(r,t) \leq 1.$$

Then $(\widetilde{X}, \widetilde{V})$ verify a SDDI, where

$$\widetilde{X}(t) := \sum_{i<j} ||x_i(t) - x_j(t)||_2 \text{ and } \widetilde{V}(t) := \frac{N(N-1)}{2} V(t).$$

Proof. Since X is only piecewise C^1 (there may be angular points at certain times when the two most distant agents change), we will consider the function $\tilde{X}$ defined for every $t \geq 0$ by

$$\widetilde{X}(t) := \sum_{i<j} ||x_i(t) - x_j(t)||_2.$$

In this way, we have that X is bounded if and only $\widetilde{X}$ is bounded. For $t \geq 0$,

$$\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t} = \sum_{i<j} \frac{1}{||x_i(t) - x_j(t)||_2} \langle x_i(t) - x_j(t), v_i(t) - v_j(t) \rangle.$$

The triangle inequality and Cauchy-Schwarz inequality leads to

$$|\frac{\mathrm{d}\tilde{X}}{\mathrm{d}t}| \leq \sum_{i<j} ||v_i(t) - v_j(t)||_2 \leq \frac{N(N-1)}{2} V(t).$$

Let $t > 0$ and $0 < \epsilon < t$, using Proposition 3.6, we get

$$\frac{V(t+\epsilon) - V(t)}{\epsilon} \leq -\frac{\mu(P_{t,t+\epsilon})}{\epsilon} V(t).$$

Because $P_{0,\epsilon} = P_{t,t+\epsilon}$ and using Proposition 3.13:

$$1 - \mu(P_{t,t+\epsilon}) = 1 - \mu(P_{0,\epsilon}) \leq C(\epsilon, \sup_{s \leq \epsilon} X(s)).$$

Even if it means reducing ϵ, we can assume

$$\sup_{s \leq \epsilon} X(s) \leq X(t) + X_0.$$

Since $r \mapsto C(t, r)$ is increasing, we get $1 - \mu(P_{t,t+\epsilon}) \leq C(\epsilon, X(t) + X_0)$. Thus

$$\frac{V(t+\epsilon) - V(t)}{\epsilon} \leq -\frac{1 - C(\epsilon, X(t) + X_0)}{\epsilon} V(t).$$

Taking $\epsilon \to 0^+$ gives

$$\frac{dV}{dt}(t) \leq -\phi(X(t))V(t), \tag{12}$$

where we noted $\phi(r) := \lim_{\epsilon \to 0^+} \frac{1 - C(\epsilon, r + X_0)}{\epsilon}$. We notice that ϕ is decreasing, so for all $t \geq 0$, $\phi(X(t)) \geq \phi(\widetilde{X}(t))$. Consequently, Equation (12) implies

$$\frac{dV}{dt}(t) \leq -\phi(\widetilde{X}(t))V(t)$$

The hypothesis $||C||_\infty \leq 1$ ensure that $\phi \geq 0$. So $(\widetilde{X}, \widetilde{V})$ verify a SDDI. ◻

3.3 Criticality of the Dobroshin's ergodicity coefficient

In section 3.2.1, we showed that the operator norm of the transition matrix associated with the semi-norm $||\cdot||_H$ was a way of characterizing the contraction of the velocity diameter. Here, we show that this operator norm is critical, in the sense that if it does not tend towards 0, then flocking is impossible. To underline the fact that it depends on the initial condition (x_0, v_0), the transition function $P : t \to P_{0,t}$ is denoted as $P_{0,t}^{x_0,v_0}$.

Lemma 3.16. *$\forall x_0 \in \mathbb{R}^{Nd}$, $\forall t \in \mathbb{R}_+$, the application $v \in \mathbb{R}^{Nd} \to P_{0,t}^{x_0,v}$ is continuous.*

Proof. Let $x_0 \in \mathbb{R}^{Nd}$ and $t \in \mathbb{R}_+$ be fixed. The matrix $P_{0,t}^{x_0,v}$ is defined as the unique solution to Equation (7). It depends continuously of the trajectory of x through Q and x depends continuously on v. ◻

Remark 3.17. *$t \to ||P_{0,t}^{x_0,v_0}||_H^{op}$ is decreasing positive, so it has a limit when $t \to +\infty$.*

Consider $H = Vect(\mathbb{1})$, subspace of $\mathbb{R}^N$. We introduce the following equivalence relation for $u, v \in \mathbb{R}^N$:

$$u \sim v \Leftrightarrow u - v \in H.$$

Lemma 3.18. *The quotient space $\mathbb{R}^N/\sim$ is a Banach space and $||\cdot||_H$ is a norm on $\mathbb{R}^N/\sim$.*

Remark 3.19. *Let $h \in H$, then for every stochastic matrix $M \in \mathbb{M}_N(\mathbb{R})$, $Mh \in H$. We also define the unit ball for the H-norm as*

$$\mathcal{B}_H(0,1) := \{v \in \mathbb{R}^N : ||v||_H \leq 1\}$$

which is a compact subset of $\mathbb{R}^N/H$.

Proposition 3.20. *Let us consider $x_0 \in \mathbb{R}^{Nd}$, assume that there exists a non-empty compact subspace B_v such that $\mathcal{B}_H(0,1) \subset B_v$ and $c > 0$ such that*

$$\forall v_0 \in B_v, \forall t \geq 0, ||P_{0,t}^{x_0,v_0}||_H^{op} \geq c$$

Then the system does not flock.

Proof. By assumption,

$$\forall v_0 \in B_v, \forall t \geq 0, \sup_{||v||_H \leq 1} ||P_{0,t}^{x_0,v_0} v||_H \geq c$$

Let $v_0 \in B_v$ be fixed. For $t_1 = 1$, we know that there exists a $v_1 \in B_v$ such that $||P_{0,t_1}^{x_0,v_0} v_1||_H \geq c$. In the same way, for $t_2 = 2$, there exists $v_2 \in B_v$ such that $||P_{0,t_2}^{x_0,v_1} v_2||_H \geq c$. Thus, we define a sequence $(t_n)_{n\in\mathbb{N}}$ by $\forall n \in \mathbb{N}, t_n := n$ and $(v_n)_{n\in\mathbb{N}} \in B_v$ such that $\forall n \in \mathbb{N}$,

$$||P_{0,t_n}^{x_0,v_n} v_{n+1}||_H \geq c.$$

Because B_v is compact, we can extract $(v_{\phi(n)})_{n\in\mathbb{N}}$ such that $v_{\phi(n)} \to v_* \in B_v$.

Let $t \in \mathbb{R}_+$, there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0, \phi(n) \geq t$. Therefore by continuity of the application $v \to ||P^{x_0,v} v||_H$ and using the fact that $t \to ||P_{0,t}^{x_0,v_{\phi(n)}}||_H$ is decreasing, we have:

$$\begin{aligned}
||P_{0,t}^{x_0,v_*} v_*||_H &= \lim_{n\to+\infty} ||P_{0,t}^{x_0,v_{\phi(n)}} v_{\phi(n)}||_H \\
&\geq \lim_{n\to+\infty} ||P_{0,\phi(n)}^{x_0,v_{\phi(n)}} v_{\phi(n)}||_H \\
&\geq c.
\end{aligned}$$

If v is a solution of the Cucker-Smale Equation (1) starting from the initial conditions (x_0, v_*), we have $||v(t)||_H = ||P_{0,t}^{x_0,v_*} v_*||_H$. Therefore, we cannot have $V(t) \underset{t\to+\infty}{\to} 0$ as $c > 0$ and there is no flocking. □

The above proposition is not applicable in practice, as we did not succeed in proving the existence of such a constant $c > 0$. We will use in section 3.4 another idea to derive non-flocking conditions.

3.4 Non-flocking conditions in the general case

For a graph with a minimal spanning tree and a communication function whose integral converges, it is always possible to find initial conditions such that no flocking occurs. Specifically, if we consider initial positions (x_0, v_0) where $V(0) > 0$, multiplying v_0 by a sufficiently large factor α will prevent flocking. Conversely, choosing a small enough factor will ensure that the system exhibits flocking behavior. The question can therefore be rephrased as follows: given an initial position x_0 and a vector v_0, by what factor α can we multiply v_0 so that starting from $(\alpha x_0, \alpha v_0)$ the system does not flock? In fact, there is a trade-off between the initial condition and the decay speed of the communication function.

Lemma 3.21. *Let us define $\psi_\alpha(r) := \psi(\alpha r)$. We note (x^α, v^α) (resp. (y^α, z^α)) the solution to the CS equation starting from initial positions $(\alpha x_0, \alpha v_0)$ (resp. (x_0, v_0)) with a communication function ψ (resp. ψ_α). We have*

$$x^\alpha = \alpha y^\alpha \text{ and } v^\alpha = \alpha z^\alpha$$

Proof. (x^α, v^α) satisfy the CS Equation (1), while (y^α, z^α) satisfy:

$$\begin{cases} \frac{\mathrm{d}\alpha y_i^\alpha}{\mathrm{d}t}(t) & = \alpha z_i^\alpha(t) \\ \frac{\mathrm{d}\alpha z_i^\alpha}{\mathrm{d}t}(t) & = \sum_{j=1}^{N} A_{i,j}\psi\big(||\alpha y_i^\alpha(t) - \alpha y_j^\alpha(t)||_2\big)(\alpha z_j^\alpha(t) - \alpha z_i^\alpha(t)). \end{cases} \tag{13}$$

We conclude with the uniqueness of the solution to the CS equation. □

The following theorem is inspired by the proof of [SYH16, Theorem 3.1] on the critical exponent of the communication function.

Theorem 3.22. *Let (x, v) be a solution of Equation (1), starting from some initial conditions $(\alpha x_0, \alpha v_0)$ where $x_0, v_0 \in \mathbb{R}^{Nd}$ and $\alpha > 0$. Let us assume that ψ satisfies (4) and that for all $i \neq j$*

$$v_i(0) \neq v_j(0) \quad \text{and} \quad \langle x_i(0) - x_j(0), v_i(0) - v_j(0)\rangle \geq 0. \tag{14}$$

We define $K := \max_{1\leq i\leq N} \sum_{j\neq i} A_{ij}$, $\bar{\psi} := \int_0^{+\infty} \psi(r)dr$ and $\rho_0 := \inf_{i\neq j} ||v_i(0) - v_j(0)||_2$. If

$$\alpha > \frac{8KV(0)}{\rho_0^2}\bar{\psi},$$

then (x, v) does not flock.

Proof. First note that by Lemma 3.21, it is as if we considered (x, v) to be a solution of the CS equation (1), starting from (x_0, v_0) and with a communication function ψ_α as defined in Lemma 3.21. We note that the first assumption is equivalent to saying that X is initially increasing. We also define $\widetilde{v}_{ij}(0) := v_i(0) - v_j(0)/||v_i(0) - v_j(0)||_2$ which is well defined since $v_i(0) \neq v_j(0)$. Let us define $T^\star = \sup\left\{T > 0 : \forall i \neq j, \ \forall t \leq T, \ \langle v_i(t) - v_j(t), \widetilde{v}_{ij}(0)\rangle > \frac{\rho_0}{2}\right\}$. By hypothesis, $\forall i \neq j$,

$$\langle v_i(0) - v_j(0), \widetilde{v}_{ij}(0)\rangle = ||v_i(0) - v_j(0)||_2 > \frac{\rho_0}{2}.$$

So we have (using the continuity of $t \to \langle v_i(t) - v_j(t), \widetilde{v}_{ij}(0)\rangle$ for all $i \neq j$), $T^\star > 0$. For all $t \leq T^\star$, Cauchy-Schwartz inequality implies $\forall i \neq j$

$$||x_i(t) - x_j(t)||_2 \geq \langle x_i(t) - x_j(t), \widetilde{v}_{i,j}\rangle.$$

Therefore,

$$||x_i(t) - x_j(t)||_2 \geq \langle x_i(0) - x_j(0), \widetilde{v}_{ij}(0)\rangle + \int_0^t \langle v_i(s) - v_j(s), \widetilde{v}_{ij}(0)\rangle ds \geq \frac{\rho_0}{2}t. \tag{15}$$

On the other hand, since the v_i are C^1, we have

$$||v_i(t) - v_i(0)||_2 \leq \int_0^t ||\frac{\mathrm{d}v_i}{dt}(s)||_2 ds.$$

A direct computation in the expression of $\frac{\mathrm{d}v_i}{dt}$ gives

$$||\frac{\mathrm{d}v_i}{dt}(s)||_2 \leq KV(0)\psi(\alpha\frac{\rho_0}{2}s),$$

where $K := \max_{1\leq i\leq N} \sum_{j\neq i} A_{ij}$. This leads to

$$||v_i(t) - v_i(0)||_2 \leq \frac{2KV(0)}{\alpha\rho_0}\bar{\psi}.$$

Now let us assume that $T^\star < +\infty$. In the one hand, by continuity of the application $t \to \langle v_i(t) - v_j(t), \widetilde{v}_{ij}(0)\rangle$, there exists $k \neq l \in \{0, \ldots, N\}$ such that

$$\langle v_k(T^*) - v_l(T^*), \widetilde{v}_{kl}(0)\rangle = \frac{\rho_0}{2}.$$

On the other hand, we have

$$\begin{aligned}\langle v_k(T^\star) - v_l(T^\star), \widetilde{v}_{kl}(0)\rangle &= \langle (v_k(0) - v_l(0)) + (v_k(T^\star) - v_k(0)) + (v_l(0) - v_l(T^\star), \widetilde{v}_{kl}(0)\rangle \\ &\geq ||v_k(0) - v_l(0)||_2 - ||v_k(T^\star) - v_k(0)||_2 - ||v_l(T^\star) - v_l(0)||_2 \\ &\geq \rho_0 - \frac{4KV(0)}{\alpha\rho_0}\bar{\psi}.\end{aligned}$$

It follows that if

$$\alpha > \frac{8KV(0)}{\rho_0^2}\bar{\psi},$$

we have

$$\rho_0 - \frac{4KV(0)}{\alpha\rho_0}\bar{\psi} > \frac{\rho_0}{2}.$$

This is impossible. In that case we have $T^* = +\infty$ which implies that the solution does not flock. □

4 A piecewise deterministic model for the collective motion of sheep

4.1 Model

Here, we consider a model inspired by the recent empirical study conducted by Gómez-Nava et al. in [GNBP22]. The study reveals that an alternating leadership model best characterizes the movement patterns of sheep. By collecting and analyzing data from a small group of sheep, researchers observed that they typically moved in a single file, with each sheep following the one ranked higher in the hierarchy. Occasionally, their movement involved pauses for grazing, during which a new leader would be chosen at random (uniformly) and initiate the next phase of their journey (see Figure 1).

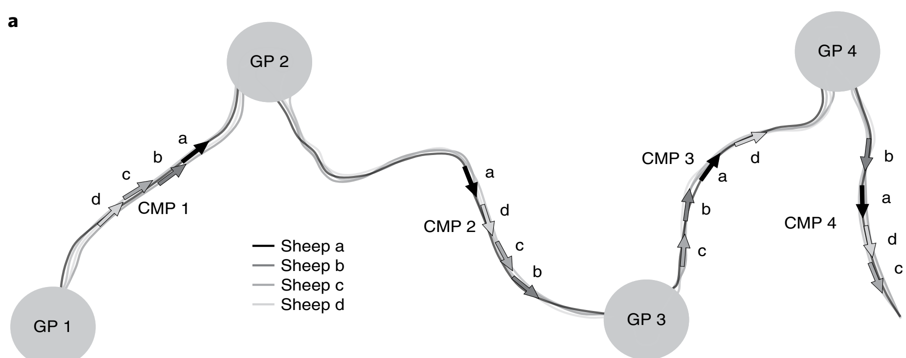


Figure 1: Intermittent collective motion, source [GNBP22].

In a modeling perspective, we construct a Piecewise Deterministic Markov Process (PDMP) inspired by [GNBP22],where the deterministic dynamics (the flow) evolves according to the Cucker-Smale equation (1) between two "stopping phases", and jumps occur on the interaction matrix A. For the sake of simplicity, we assume that after each stop, the sheep return to the velocity they had before stopping. More formally, let $(T_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that $\forall i \in \mathbb{N}$, $T_i \sim \mathcal{E}(\lambda)$, were $\lambda > 0$. We assume here that the jump rate λ is constant, but possible generalizations could be to consider jump rates that depend on the current state, the distance covered since the previous stop, etc. We define the jumping times for $n \geq 1$, as a Poisson process $S_n = T_1 + \cdots T_n$. Starting from the initial conditions $(x_i(0), v_i(0))_{1\leq i\leq n} \in \mathbb{R}^d \times \mathbb{R}^d$ and a matrix $A^{(1)}$. For $t \in [0, S_1]$, the model follows the deterministic dynamic (1). At time $t = S_1$, the matrix $A^{(1)}$ jumps to another matrix $A^{(2)}$ with probability one. Then, the dynamics follows (1) again, with $A^{(2)}$ and initial conditions $x(S_1^-)$ =(etc. When using interaction matrices of the form $0 \to 1 \to \cdots, \to N$ and its "opposite" $N \to N-1 \to \cdots, \to 0$ as $A^{(1)}$ and $A^{(2)}$ with a jump rate λ large enough, we managed to find some cases where there was flocking for both matrices but not when switching from one to the other quickly enough (see Figure 3b). More formally, let $(T_i)_{i\in\mathbb{N}}$ be a sequence of i.i.d. random variables such that $\forall i \in \mathbb{N}$, $T_i \sim \mathcal{E}(\lambda)$, were $\lambda > 0$. We assume here that the jump rate λ is constant, but possible generalizations could be to consider jump rates that depend on the current state, the distance covered since the previous stop, etc. We define the jumping times for $n \geq 1$, as a Poisson process $S_n = T_1 + \cdots T_n$. Starting from the initial conditions $(x_i(0), v_i(0))_{1\leq i\leq n} \in \mathbb{R}^d \times \mathbb{R}^d$ and a matrix $A^{(1)}$. For $t \in [0, S_1]$, the model follows the deterministic dynamic (1). At time $t = S_1$, the matrix $A^{(1)}$ jumps to another matrix $A^{(2)}$ with probability one. Then, the dynamics follows (1) again, with $A^{(2)}$, initial conditions $x(S_1^-) = x(S_1^+)$ and $v(S_1^-) = v(S_1^+)$ and so on. When using interaction matrices of the form $0 \to 1 \to \cdots, \to N$ and its "opposite" $N \to N-1 \to \cdots, \to 0$ as $A^{(1)}$ and $A^{(2)}$ with a jump rate λ large enough, we managed to find some cases where there was flocking for both matrices but not when switching from one to the other quickly enough (see Figure 3b).

Remark 4.1. *This type of PDMP model in which the interaction matrix jumps randomly was studied in [DHJK20] in the case where the interaction graph does not necessarily have an unique spanning tree. They showed that under certain conditions on the transition probabilities and if the chain jumps quickly enough, we could have flocking almost surely even if there was no flocking for each of these matrices.*

More formally, let $(I_t)_{t\geq 0}$ be a jump process with values in $\mathcal{S}_I = \{1, 2\}$ of intensity $\lambda > 0$. The model now reads as follows:

$$\begin{cases} \frac{dx_i}{dt}(t) &= v_i(t) \\ \frac{dv_i}{dt}(t) &= \sum_{j=1}^{N} Q_t^{(I_t)}(i,j)(v_j(t) - v_i(t)), \end{cases} \tag{16}$$

where $Q_t^{(I_t)}(i,j) := A^{(I_t)}(i,j)\psi(||x_i(t) - x_j(t)||_2)$.
For $k \in \mathcal{S}_I$, let $P^{(k)} : (s,t) \mapsto P_{s,t}^{(k)} \in \mathbb{M}_N(\mathbb{R})$ be the solution of

$$\begin{cases} \partial_t P_{s,t}^{(k)} &= Q_t^{(k)} P_{s,t}^{(k)} \\ P_{s,s}^{(k)} &= I_N. \end{cases}$$

These functions satisfy the "semi group property". Let $k \in \mathcal{S}_I$, for v^k solution of (1) with $Q_t = Q_t^{(k)}$, we have $\forall s \leq u \leq t$,

$$v^k(t) = P_{s,t}^{(k)} v^k(s) \text{ and } P_{s,t}^{(k)} = P_{s,u}^{(k)} P_{u,t}^{(k)}.$$

We define for $t \geq 0$, $n_t := \sup\{n \in \mathbb{N} : S_n \leq t\}$. Let $(x(t), v(t))_{t\geq 0}$ be a solution of Equation (16), then we have for all $t \in [S_{n_t}, S_{n_t+1}]$,

$$v(t) = P_{S_0,S_1}^{(I_0)} P_{S_1,S_2}^{(I_1)} \ldots P_{S_{n_t-1},S_{n_t}}^{(I_{n_t-1})} P_{S_{n_t},t}^{(I_{n_t})} v(0). \tag{17}$$

4.2 Two agents case

Following the same framework as in Section 3.4, we focus our study on the case of two agents as we have a complete description of the behavior in the deterministic model. In this section, we will focus on the case of two agents, evolving according to following Equation:

$$\begin{cases} \frac{\mathrm{d}x_1}{\mathrm{d}t}(t) &= v_1(t) \\ \frac{\mathrm{d}x_2}{\mathrm{d}t}(t) &= v_2(t) \\ \frac{\mathrm{d}v_1}{\mathrm{d}t}(t) &= A_1 \mathbb{1}_{I_t=1}\psi(||x_1(t) - x_2(t)||_2)(v_2(t) - v_1(t)) \\ \frac{\mathrm{d}v_2}{\mathrm{d}t}(t) &= A_2 \mathbb{1}_{I_t=2}\psi(||x_1(t) - x_2(t)||_2)(v_1(t) - v_2(t)). \end{cases} \tag{18}$$

Where A_1 and A_2 are two positive real numbers. In the dynamic described by the above system, we chose to consider interactions matrices of the form

$$A^{(1)} := \begin{bmatrix} -A_1 & A_1 \\ 0 & 0 \end{bmatrix} \text{ and } A^{(2)} := \begin{bmatrix} 0 & 0 \\ A_2 & -A_2 \end{bmatrix} \tag{19}$$

for the sake of simplicity. This corresponds to a situation where agents follow each other alternately.

Theorem 4.2. *If (x_1, x_2, v_1, v_2) is a solution to (18), then we can derive flocking conditions:*

- *If $V_0 < (A_1 \wedge A_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$ there is flocking almost surely.*
- *If $V_0 \geq (A_1 \vee A_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$ there is no flocking almost surely*
- *If $(A_1 \wedge A_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr \leq V_0 < (A_1 \vee A_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$ there is flocking with a probability strictly between 0 and 1.*

Proof. As we are interested in the diameter in position and velocity, we introduce $x = x_1 - x_2$ and $v = v_1 - v_2$, which verify the following equations:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t}(t) &= v(t) \\ \frac{\mathrm{d}v}{\mathrm{d}t}(t) &= -K_t\psi(||x(t)||_2)v(t) \end{cases} \tag{20}$$

where $K_t := A_{I_t}$. We also denote $\phi(r,t) := K_t\psi(r)$. We notice that $\forall t \geq 0$

$$\frac{\mathrm{d}\langle v(t), v_0\rangle}{\mathrm{d}t} = -\phi(||x(t)||_2, t)\langle v(t), v_0\rangle \text{ and } \frac{\mathrm{d}||v(t)||_2}{\mathrm{d}t} = -\phi(||x(t)||_2, t)||v(t)||_2. \tag{21}$$

So we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\langle v_0, v(t)\rangle}{||v_0||_2||v(t)||_2} = 0,$$

and then $\langle v_0, v(t)\rangle = ||v_0||_2||v(t)||_2$. Consequently, $v(t) = V(t)\bar{v}_0$ and $\forall t \geq 0$

$$x(t) = x_0 + \bar{v}_0\int_0^t V(s)ds.$$

In this simple case, we have for all $t \geq 0$, $X(t) = ||x_1(t) - x_2(t)||_2$ and $V(t) = ||v_1(t) - v_2(t)||_2$. We will show that (X, V) satisfy some differential equality that will allow us to conclude about their long time behavior.

Colinear case. Suppose that $\langle x_0, v_0\rangle = ||x_0||_2||v_0||_2$, i.e. the vectors x_0 and v_0 are colinear. We have $\forall t \geq 0$

$$x(t) = X(t)\bar{v}_0 \text{ and } X(t) = X_0 + \int_0^t V(s)ds. \tag{22}$$

So by (21) and (22), (X, V) satisfies the following system of dissipative differential equality :

$$\begin{cases} \frac{\mathrm{d}X}{\mathrm{d}t}(t) &= V(t) \\ \frac{\mathrm{d}V}{\mathrm{d}t}(t) &= -\phi(X(t), t)V(t). \end{cases} \tag{23}$$

Since $\frac{\mathrm{d}X}{\mathrm{d}t} = V \geq 0$, X is an increasing function in time.

According to Lemma 3.8, we have $V \neq 0$ (we assume here that $v_0 \neq 0$, because otherwise there's nothing to show), so X is strictly increasing. Let $t \in [S_{n_t}, S_{n_t+1}]$, Equation (23) implies that

$$\begin{aligned} V(t) &= V_0 - \int_0^t K_r\psi(X(r))V(r)dr \\ &= V_0 - \sum_{i=0}^{n_t} K_{S_i}\int_{S_i}^{S_{i+1}} \psi(X(r))V(r)dr - K_{S_{n_t}}\int_{S_{n_t}}^t \psi(X(r))dr \\ &= V_0 - \sum_{i=0}^{n_t} K_{S_i}\int_{X(S_i)}^{X(S_{i+1})} \psi(s)ds - K_{S_{n_t}}\int_{X(S_{n_t})}^{X(t)} \psi(s)ds \text{ by change of variables} \\ &= V_0 - \int_{X_0}^{X(t)} \widetilde{K}(s)\psi(s)ds, \end{aligned} \tag{24}$$

where the function $\widetilde{K}$ is defined as $\widetilde{K}(s) := K_i$, with $i = \sup\{S_i : i \in \mathbb{N}, X(S_i) \leq s\}$ for $s \leq \sup_{t\geq 0} X(t)$.

(i) If $V_0 < (K_1 \wedge K_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$. Then, there exists $X_M \geq X_0$ such that

$$V_0 = (K_1 \wedge K_2)\int_{X_0}^{X_M} \psi(r)dr,$$

which leads to

$$\begin{aligned} V(t) &\leq V_0 - \int_{X_0}^{X(t)} (K_1 \wedge K_2)\psi(s)ds \\ &= (K_1 \wedge K_2)\int_{X(t)}^{X_M} \psi(s)ds. \end{aligned}$$

Since $V \geq 0$, we necessarily have $X \leq X_M$. Thus using equation (23), and the fact that ψ is decreasing, we deduce

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -K_t\psi(X_M)V(t).$$

Then, from Grönwall's lemma, it follows that

$$V(t) \leq V_0 \mathrm{e}^{-\psi(X_M)\int_0^t K_s ds},$$

which allows us to conclude that the flocking condition are verified. Indeed, we have $V(t) \underset{t\to+\infty}{\to} 0$ and since V is integrable, we obviously deduce that X is bounded (21).

(ii) if $V_0 > (K_1 \vee K_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$, by Equation (24), we deduce

$$\begin{aligned} V(t) &\geq V_0 - (K_1 \vee K_2)\int_{X_0}^{X(t)} \psi(r)dr \\ &> V_0 - (K_1 \vee K_2)\int_{X_0}^{+\infty} \psi(r)dr > 0. \end{aligned}$$

Therefore we cannot have $V(t) \underset{t\to+\infty}{\to} 0$, and the system does not flock.

(ii) if $V_0 = (K_1 \vee K_2)\displaystyle\int_{X_0}^{+\infty} \psi(r)dr$, then by Equation (24), we easily deduce

$$V(t) \geq (K_1 \vee K_2)\int_{X(t)}^{+\infty} \psi(r)dr.$$

If X is bounded, the right-hand term is strictly greater than 0 so we cannot have $V(t) \to 0$ and X bounded, i.e. the system does not flock.

We now turn our attention to the case where V_0 is between these two bounds. Without loss of generality, we can assume that $K_1 \leq K_2$. If $K_1\displaystyle\int_{X_0}^{+\infty} \psi(s)ds \leq V_0 < K_2\int_{X_0}^{+\infty} \psi(s)ds$, let us show that both situations, flocking and non-flocking, can occur with strictly positive probabilities.
We first assume that $I_0 = 2$. Let us show that the system can flock with a strictly positive probability. According to the case (i), there exists $X_M \geq X_0$ such that for $t \in [S_0, S_1]$,

$$V(t) \leq V_0 \mathrm{e}^{-\psi(X_M)K_2 t} \text{ and } X(S_1) \leq X_M. \tag{25}$$

Using case (i), and Equation (25), we have:

$$\begin{aligned}\mathbb{P}(\{\text{the system flocks}\}) &\geq \mathbb{P}\left(V(S_1) \leq \int_{X(S_1)}^{+\infty} K_1\psi(r)\,dr\right)\\ &\geq \mathbb{P}\left(V(S_1) \leq \int_{X_M}^{+\infty} K_1\psi(r)\,dr\right)\\ &\geq \mathbb{P}\left(S_1 \geq -\frac{1}{K_2\psi(X_M)}\ln\left(\frac{1}{V_0}\int_{X_M}^{+\infty} K_1\psi(r)\,dr\right)\right) > 0.\end{aligned}$$

Let's show that the system can also not flock with a strictly positive probability (i.e. the probability of flocking is strictly less than one). By continuity of X, there exists $t_0 \geq 0$ such that

$$\int_{X(t_0)}^{+\infty} K_1\psi(r)\,dr \leq V(t_0) \leq \int_{X(t_0)}^{+\infty} K_2\psi(r)\,dr.$$

By definition of the jump-time process S, $\mathbb{P}(S_1 \leq t_0) = 1 - \mathrm{e}^{-\lambda t_0} > 0$. For $t \in [S_1, S_2], X(t) \geq X(S_1) + ct$ for some $c \geq 0$ according to Equation (15). In particular,

$$X(S_2) \geq X(S_1) + cS_2. \tag{26}$$

Moreover, since $\displaystyle\int_{X(S_1)+ct}^{+\infty} K_2\psi(r)dr \underset{t\to+\infty}{\rightarrow} 0$, there exists almost surely $t_1 \geq S_1$ such that

$$\int_{X(S_1)+ct_1}^{+\infty} K_2\psi(r)dr \leq V(S_2). \tag{27}$$

We also have $t_1 \leq S_2$ with probability $1 - e^{-\lambda t_1}$ so Equation (26) and (27) allow us to conclude that with a strictly positive probability

$$\int_{X(S_2)}^{+\infty} (K_1 \vee K_2)\psi(r)dr < V(S_2).$$

According to case (ii) above, there is no flocking. If we assumed $I_0 = 1$, it is the same idea. Indeed, for $t \in [0, S_1]$, we are not in a flocking situation (case (ii)) as for $t \in [0, S_1]$, the condition

$$V(t) < \int_{X(t)}^{+\infty} (K_1 \vee K_1)\psi(r)dr.$$

will not be satisfied for any $tin[0, S_1]$ and once the first jump occurs, we end up in the situation described above.

Non colinear case: we no longer assume that x_0 and v_0 are colinear.

Let us define $\widetilde{X}_0 = \langle \bar{v}_0, x_0\rangle$, $Y_0 = \sqrt{||x_0||_2^2 - \widetilde{X}_0^2}$, $\bar{w}_0 = \frac{x_0 - \widetilde{X}_0\bar{v}_0}{Y_0}$.
It is easy to check that

$$||\bar{w}_0||_2 = 1, \langle \bar{v}_0, \bar{w}_0\rangle = 0 \text{ and } \langle x_0, \bar{w}_0\rangle \geq 0.$$

A simple computation gives $x(t) = \left(\widetilde{X}_0 + \displaystyle\int_0^t V(s)ds\right)\bar{v}_0 + Y_0\bar{w}_0$. Let us define $\widetilde{X}(t) := \langle \bar{v}_0, x(t)\rangle = \widetilde{X}_0 + \displaystyle\int_0^t V(s)ds$. Since $\bar{v}_0$ and $\bar{w}_0$ are orthogonal, we have $||x(t)||_2^2 = Y_0^2 + \widetilde{X}(t)^2$ and so $(\widetilde{X}, V)$ satisfies:

$$\begin{cases}\frac{\mathrm{d}\widetilde{X}}{\mathrm{d}t}(t) &= V(t)\\ \frac{\mathrm{d}V}{\mathrm{d}t}(t) &= -\phi_{Y_0}(\widetilde{X}(t), t)V(t),\end{cases}$$

where $\phi_{Y_0}(\widetilde{X}(t), t) = A_t\psi_{Y_0}(\widetilde{X}(t))$. We note that (X, V) flocks if and only if $(\widetilde{X}, V)$ is flocks, and since the latter couple satisfies a system analogous to (23), it suffices to apply to $(\widetilde{X}, V)$ the same proof as that for (X, V) in the colinear case.

□

4.3 Simulations

As shown in Theorem 4.2, when $V_0 \in [B_1, B_2]$ where $B_1 := A_1 \int_{X_0}^{+\infty} \psi(r)\, dr < B_2 := A_2 \int_{X_0}^{+\infty} \psi(r)\, dr$, there exists a probability strictly between 0 and 1 that the solution to Equation (18) will flock. We have run simulations to estimate this probability for various values of V_0 within this interval. The simulations were performed with fixed initial positions for the two agents $(x_1(0), x_2(0))$, a jump rate of $\lambda = 0.8$, and two matrices $A^{(1)}$ and $A^{(2)}$ defined as in (19). Without loss of generality, we assumed that $v_1(0) = 0$. Thus, for every value of $V_0 = \|v_1(0) - v_2(0)\|_2$, it suffices to choose an angle $\theta \in [0, 2\pi]$ to fully determine the initial conditions. We chose this angle at random between 0 and 2π. For each value of V_0, we simulated the trajectory $n = 200$ times and checked when the solution flocked. The flocking frequency as a function of V_0 is plotted in Figure 2.

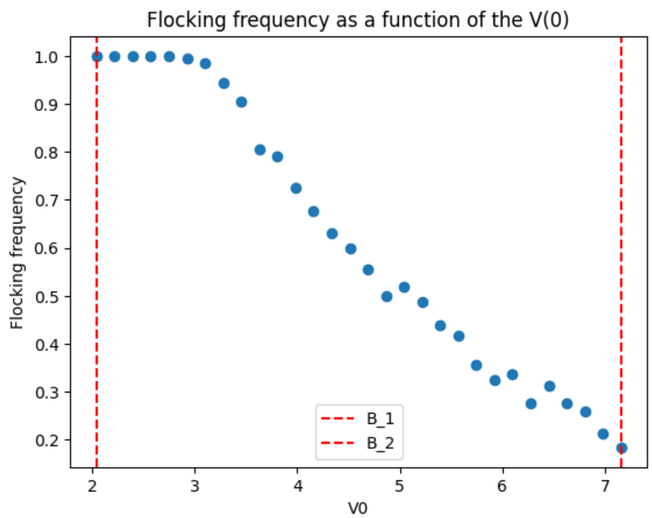


Figure 2: Flocking frequency as a function of V_0.

For values of V_0 close to the lower bound B_1, the frequency of flocking trajectories is close to 1 and continuously decreases to 0 as V_0 approaches B_2.

4.4 Flocking in the general case

We now consider the model in the general case of $N \geq 3$ agents defined in Section 4.1, by Equation (16). We have a Proposition analogous to Proposition 3.12 in the case of switching :

Proposition 4.3. *Let us suppose that there exists $C : \mathbb{R}_+^2 \to \mathbb{R}_+$, such that:*

1. *$\forall t \geq 0, r \mapsto C(t, r)$ is increasing,*
2. *$\forall k \in \mathcal{S}_I, \forall t \geq 0, 1 - \mu\left(P_{0,t}^{(k)}\right) \leq C(t, \sup_{s \leq t} X(s))$,*

where μ is the Dobroshin's coefficient defined in theorem 3.4. Then finding $r_0 \geq X_0$ such that

$$r_0 - X_0 > \sum_{i \geq 0} U_i(r_0) \ \ a.s. \tag{28}$$

where $U_i(r_0) = V(S_i) \int_{S_i}^{S_{i+1}} C(s - S_i, r_0) ds$ leads to flocking.

Proof. For $i, j \in \{1, \dots N\}, t \geq 0$, by definition of the model:

$$x_i(t) - x_j(t) = x_i(0) - x_j(0) + \int_0^t v_i(t) - v_j(t) dt.$$

Which leads to $X(t) \leq X_0 + \int_0^t V(s)ds$.
Let $r_0 \geq X_0$ and $\tau := \sup\{t \geq 0 \mid \sup_{s\leq t} X(s) \leq r_0\}$. Assume that $\tau < +\infty$. By continuity of X, we have $X(\tau) = r_0$, so by the above inequality we deduce

$$\begin{aligned} r_0 - X_0 &\leq \int_0^\tau V(s)ds \leq \int_0^{+\infty} V(s)ds \\ &\leq \sum_{i=0}^{+\infty} \int_{S_i}^{S_i+1} V(s)ds. \end{aligned}$$

Using remark 3.2, Proposition 3.6 and hypothesis (1) we have that for all $n \in \mathbb{N}$, $t \in [S_n, S_{n+1}]$,

$$\begin{aligned} V(t) &\leq (1 - \mu(P_{S_n,t}))V(S_n) = (1 - \mu(P_{0,t-S_n}))V(S_n) \\ &\leq C(t - S_n, \sup_{s\leq t-S_n} X(s))V(S_n). \end{aligned}$$

In addition, since $\forall t \geq 0, r_0 \geq X(t)$, hypothesis (2) implies that $V(t) \leq C(t - S_n, r_0)V(S_n)$ for $t \in [S_n, S_{n+1}]$. Consequently, the two inequalities above leads to

$$\begin{aligned} r - X_0 &\leq \sum_{i=0}^{+\infty} V(S_i) \int_{S_i}^{S_{i+1}} C(s - S_i, r_0)ds \\ &= \sum_{i\geq 0} U_i(r_0). \end{aligned}$$

This is impossible because of Equation (28). Thus we have $\tau = +\infty$. Being able to find r_0 such that equation (28) holds implies that $\sum_i U_i(r_0) < +\infty$ a.s. so we also have $\int_0^{+\infty} V(s)ds < +\infty$ a.s.
In the switching case, V is also a decreasing function of time (Proposition 3.7). In fact, between each jump time, $s \mapsto V(s)$ is decreasing, so by the continuity of V, it is decreasing on $\mathbb{R}_+$. Since V is positive and of finite integral, it admits a limit in $+\infty$ which must be 0. The diameter X is bounded by r_0. □

In our examples, it is sufficient to show that there exists $r_0 \geq X_0$ such that $\sum_i U_i(r_0) < +\infty$ a.s to prove flocking. It is therefore sufficient to show that $\mathbb{E}(\sum_i)U_i(r_0)) < +\infty$.

Remark 4.4. *In order to find initial conditions such that the system flocks in the case of a scrambling matrix $Ain\mathbb{M}_N(\mathbb{R})$. Cotil uses in [Cot23] the function C_k, $k \in \mathcal{S}_I$ defined for all $t, r \geq 0$ by*

$$C_k(t, r) = \mathbb{P}\Big(\Gamma_H \geq A_*^{(k)}\psi(r)t\Big)$$

where Γ_H follows a gamma distribution of parameter shape H and scale 1, with $H \in \mathbb{N}$ well chosen ([[Cot23] Theorem 3.7]) and $A_* := \inf_{i>1} \sum_{j\neq i} A_{i,j}$.
In the switching case, we can use Proposition 4.3, with the function $\bar{C}$ defined as

$$\bar{C}(t, r) := C_0(t, r_0) \vee C_1(t, r_0) = \mathbb{P}\Big(\Gamma_H \geq (A_*^{(0)} \wedge A_*^{(1)})\psi(r_0)t\Big),$$

where C_0 C_1 are defined above.

Remark 4.5. *Proposition 3.6, 3.12 and the above remark implies that for all $n \in \mathbb{N}$:*

$$V(S_n) \leq \prod_{k=0}^{n} \big(1 - \mu(P_{0,S_{k+1}-S_k})\big)V(0) \leq \prod_{k=0}^{n} \bar{C}(T_k, r_0)V(0).$$

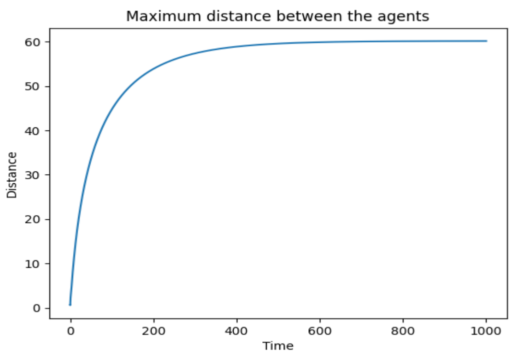
By definition, $S_{k+1} - S_k = T_k \sim \mathcal{E}(\lambda)$. Taking the expectation, we get the independence of $(T_k)_{k\geq 0}$, for all $n \in \mathbb{N}$,

$$\mathbb{E}(V(S_n)) \leq V(0)\mathbb{E}\big(\bar{C}(T, r_0)\big)^n.$$

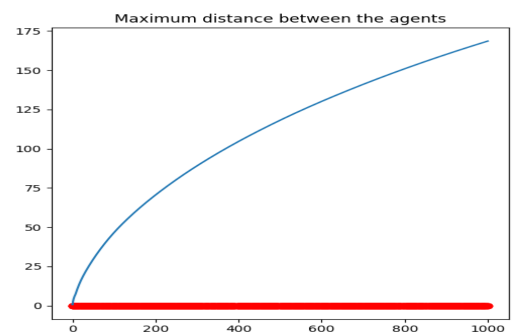
Where $\bar{C}$ is defined above and T is the inter jump random variable following an exponential law of parameter λ. So if $(\mathbb{E}\big(\bar{C}(T, r_0)\big)) < 1$ there is convergence of $\sum_i U_i(r_0)$ almost surely and finding a $r_0 \geq X_0$ such that (28) holds is possible. In the case where $C(t, \cdot) < 1$ for some $t \geq 0$, this will always be the case.

5 Simulations

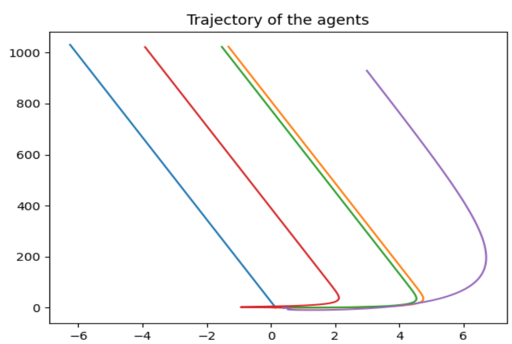
When performing simulations we managed to find initial conditions (X_0, V_0) and a jump intensity parameter $\lambda > 0$ such that there is flocking for both A_0 and A_1 (see Figure 3a) but not when switching from A_0 to A_1 with intensity λ (see Figure 3b).
However, whether it is for A_0, A_1, there is no r_0 such that condition (28) holds. This is not surprising, as this condition is far too restrictive and only captures a small number of cases where there is flocking. To find such an r_0, we had to change the initial conditions so that V_0 was small enough.



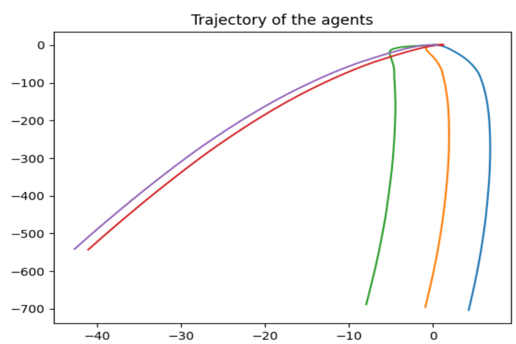
(a) Diameters of positions in the case of a fixed interaction matrix A_0



(b) Diameters of positions when switching between A_0 and A_1 and the jump times (red dots)



(a) Trajectories of the agents in the case of a fixed interaction matrix A_0



(b) Trajectories of the agents when switching between interaction matrices A_0 and A_1

6 Discussion

In this report, we extended some results presented by A. Cotil in [Cot23]. Specifically, we derived both necessary and sufficient conditions for flocking in the case of two agents and found a sufficient condition for non-flocking in the general case. We also established the criticality of the Dobroshin ergodicity coefficient. However, some questions remain unanswered. For instance, the conditions we impose on the initial conditions to observe flocking in Proposition 3.12 imply that V is of finite integral, from which we deduce it tends towards 0 (since it has a limit). But what is shown in this proposition is stronger than just $V \underset{t\to+\infty}{\to} 0$. We do not know whether flocking necessarily implies that V is of finite integral. Furthermore, we proposed a stochastic extension of the CS model on a weighted directed graph and provided results analogous to those presented in [Cot23]. However, we did not extend Theorem 3.22 to switching topologies. In contrast, [DHJK20] found sufficient conditions for flocking of this model when the interaction graph is poorly connected (i.e., it does not have a spanning tree) but can still achieve flocking almost surely if switching occurs frequently enough. Another potential question would be to consider the dynamics governed by the following system of ordinary differential equations:

$$\frac{dx_i(t)}{dt} = v_i(t), \quad \frac{dv_i(t)}{dt} = \sum_{j=1}^{N}\sum_{q=1}^{d} A_{ij}B_{pq}\psi\left(\|x_j(t)-x_i(t)\|_2^2\right)(v_j(t)-v_i(t)), \tag{29}$$

where $B \in \mathbb{R}^{d\times d}$ and $A \in \mathbb{R}_+^{N\times N}$. This model accommodates a broad range of communication protocols, where factors other than the spatial state influence the alignment linearly. In this generalized framework, an agent updates its velocity based on the differential velocity relative to other agents across all coordinates. If we now consider the matrix B as randomly switching, there could be counter-intuitive results. For instance, the system might exhibit flocking behavior for matrices $B^{(1)}$ and $B^{(2)}$ individually, but not when switching between them at a sufficient rate. This phenomenon is similar to the one discussed in [BlBMZ14]. It is possible that this behavior could also apply to the model described by Equation (29) as B is assumed to have positive eigenvalues (which is not the case for matrix A). Another perspective could be to study the following stochastic alignment model. Contrary to what many models assume, interaction between agents does not depend on the metric distance but rather on the topological distance. In fact, it was discovered that each bird interacts on average with a fixed number of neighbors (six to seven), rather than with all neighbors within a fixed metric distance [CDH19]. To this end, the term $\psi(||x_i(t)-x_j(t)||_2)$ in the original Cucker-Smale model (Equations (1)) was replaced in [Has13] by $\psi\left(\sum_i \mathbb{1}_{|x_l(t)-x_i(t)|<|x_i(t)-x_j(t)|}\right)$ to have interaction only depending on their proximity or their ranks. Following this idea, Blanchet and Degond introduced in [BD17] the following model:

- Between two jump times, agents follow straight paths with constant velocity, i.e., $dx_i(t) = v_i(t)dt$ and $dv_i(t) = 0$.
- Jumps occur according to a Poisson process $N(t)$ with intensity (rate) $\lambda(N)$, i.e., times between jumps are independent, identically distributed according to an exponential distribution with parameter $\lambda(N)$. When the clock of the Poisson process rings (i.e., at each jump time), an agent is chosen at random (i.e., uniformly with probability 1/N).
- When it "jumps," agent i chooses its partner j according to a probability $\pi_{i,j}^N$ to be described below, and then (x_i, v_i) is changed into (x_i, v_j), i.e., i immediately aligns its velocity with j.

This model presents an alternative by focusing on topological rather than metric distances and has connections with other famous stochastic models, such as the Moran process in genetics.

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