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On the limit law of the superdiffusive elephant random walk*

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Abstract

When the memory parameter of the elephant random walk is above a critical threshold, the process becomes superdiffusive and, once suitably normalised, converges to a non-Gaussian random variable. In a recent paper by the first three authors, it was shown that this limit variable has a density and that the associated moments satisfy a nonlinear recurrence relation. In this work, we exploit this recurrence to derive an asymptotic expansion of the moments and the asymptotic behaviour of the density at infinity. In particular, we show that an asymmetry in the distribution of the first step of the random walk leads to an asymmetry of the tails of the limit variable. These results follow from a new, explicit expression of the Stieltjes transformation of the moments in terms of special functions such as hypergeometric series and incomplete beta integrals. We also obtain other results about the random variable, such as unimodality and, for certain values of the memory parameter, log-concavity.

Keywords: algebraic ordinary differential equation; elephant random walk; incomplete beta function; log-concavity; Mittag-Leffler function; strong Tauberian theorem; superdiffusive limit; unimodality.
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1 Introduction and statement of the results

The one-dimensional elephant random walk (ERW) was introduced in 2004 by Schütz and Trimper [26] to see how memory could induce anomalous diffusion in random walk processes. It turned out that the ERW is always at least diffusive; nevertheless, the simple definition of the process, together with the underlying depth of the results, has led to a great deal of interest from mathematicians over the last two decades.

The ERW process $(S_n)_{n\geqslant 0}$ is defined as follows. We denote its successive steps by $(X_n)_{n\geqslant 0}$. The elephant starts at the origin at time zero: $S_0 = 0$. For the first step X_1, the elephant moves one step to the right with probability q or one step to the left with probability $1-q$, for some q in $[0,1]$. The next steps are performed by uniformly choosing an integer k from the previous times. Then the elephant moves exactly in the same direction as at time k with probability $p \in [0,1]$, or in the opposite direction with probability $1-p$. In other words, defining for all $n \geqslant 1$,

$$X_{n+1} = \begin{cases} +X_{\mathcal{U}(n)} & \text{with probability } p, \\ -X_{\mathcal{U}(n)} & \text{with probability } 1-p, \end{cases}$$

with $\mathcal{U}(n)$ an independent uniform random variable on $\{1,\dots,n\}$, the position of the ERW at time $n+1$ is

$$S_{n+1} = S_n + X_{n+1}.$$

The probability q is called the first step parameter and p the memory parameter of the ERW. An ERW trajectory is sampled in Figure 1.

A wide range of literature is now available on the ERW and its extensions, see for instance [2, 7, 11, 10, 12, 22]. The behaviour of the process, in particular its dependency on the value of p with respect to the critical value $3/4$, is now well understood. In the diffusive regime $p < 3/4$ and the critical regime $p = 3/4$, a strong law of large numbers and a central limit theorem for the position, properly normalized, were established, see [2, 7, 8, 26] and the more recent contributions [5, 9, 14, 18, 28]. The main change between the two regimes is the rate of the associated convergences.

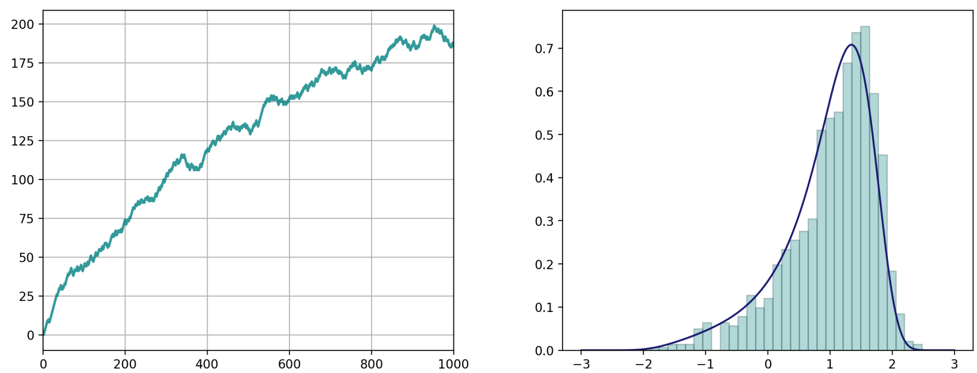

Figure 1: Left display: a trajectory of the elephant random walk in the superdiffusive regime until time $n = 1000$, with $p = 0.92$ and $q = 1$. Right display: a simulation of the density of L_1 for the same parameters.

Introduce $a := 2p-1$. In this paper, we will focus on the superdiffusive regime $a > 1/2$ (i.e., $p > 3/4$), which is more intriguing. It has been established that

$$\lim_{n\to\infty} \frac{S_n}{n^a} = L_q \quad \text{a.s.}$$

where L_q is a non-degenerate, non-Gaussian random variable, see [2, 4, 7]. Moreover, the fluctuations of the ERW around its limit L_q are Gaussian by [21]. These results were established thanks to a martingale approach.

While the limit variable L_q has remained mysterious for some time, the recent paper [20] established several properties of it, such as the existence of a density, a recursive formula for the moments, the finiteness of the moment-generating function and the moment problem. This was achieved by reformulating the ERW in terms of urn processes together with fixed point equations.

In the present work, we pursue this line of research and obtain fine properties of the distribution, via other methods. We discard the case $a = 1$ which is of no interest, since then $L_q \overset{a.s.}{=} X_1$ is the first step of the elephant random walk. We focus the study on the distribution of L_1, the limit of the superdiffusive ERW random variable with parameter $a \in (1/2, 1)$ and first step parameter $q = 1$. As observed in [20, Rem. 2.3], there is a simple relation between the distribution of L_1 and the general asymptotic variable L_q:

$$L_q \overset{d}{=} (2\xi_q - 1)L_1, \tag{1.1}$$

where ξ_q is a Bernoulli variable with parameter q, independent of L_1. Our first result is the following.

Theorem 1.1. *The random variable L_1 is unimodal.*

If we set φ for the density function of L_1 on $\mathbb{R}$, which is known to exist thanks to [20, Thm 1.3], let us recall that the unimodality of L_1 means that this function has a unique local maximum. Our method to obtain this important property is to show first the discrete unimodality of S_n itself on $\mathbb{Z}$ for all n, and then take the renormalized limit defining L_1. This discrete unimodality is obtained by induction, using the (non-homogeneous) Markovian character of the ERW. A classical refinement of unimodality for sequences or functions is the log-concavity, which is also known as a strong unimodality property since it preserves unimodality under additive convolution. For more details on this topic, see for example the survey papers [25, 27]. In the present paper, we are able to characterize the log-concavity of the ERW for all n in terms of a certain threshold parameter a_0 – see Proposition 2.2, which implies that the function φ is log-concave on $\mathbb{R}$ for all $a \in (1/2, a_0]$. We also believe that the ERW becomes log-concave for all $a \in (1/2, 1)$ and n large enough, but this non-trivial problem, which involves complicated polynomials, still eludes us.

Our next results give the precise asymptotic behaviour of φ at $\pm\infty$. These estimates involve a curious parameter ρ_a, which has not appeared yet in the ERW literature:

$$\rho_a := \left(\frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2} + \frac{1}{2a}\right) \Gamma\left(1 - \frac{1}{2a}\right)\right)^a = \left(\frac{\mathrm{B}\left(\frac{1}{2} + \frac{1}{2a}, 1 - \frac{1}{2a}\right)}{2}\right)^a. \tag{1.2}$$

As proved in Proposition 3.5, the function $a \mapsto \rho_a$ is decreasing on $(1/2, 1)$ from $+\infty$ to 1, see also Figure 2.

Theorem 1.2. *One has*

$$\varphi(x) \sim c_a \, x^{\frac{2a-1}{2(1-a)}} \, e^{-(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}}}$$

as $x \to \infty$, with c_a an explicit constant given in (4.1).

For $a \in (1/2, 1)$, we have $\frac{1}{1-a} > 2$ and we hence recover the fact, first pointed out by Bercu in [3, Rem. 3.2], then refined in [20, Cor. 2.12], that the distribution of L_1 is sub-Gaussian.

Theorem 1.3. *One has*

$$\varphi(-x) \sim \widehat{c}_a \, x^{\frac{2a^2-3a-1}{2(1-a^2)}} \, e^{-(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}}}$$

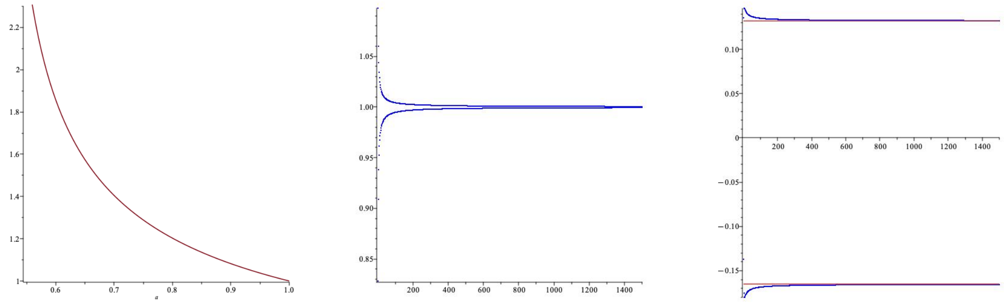

Figure 2: Left display: graph of the exponential growth ρ_a on $(\frac{1}{2}, 1)$. The function ρ_a is decreasing on $(\frac{1}{2}, 1)$, as will be proved in Proposition 3.5. Middle display: the ratio $m_n/(\frac{2a}{a+1}\rho_a^n)$ converges to 1 as $n \to +\infty$, according to Theorem 3.1 (computations made for $a = \frac{2}{3}$, using the exact recurrence relation (3.5)). Right display: the ratio $n^{\frac{2a}{a+1}}(m_n - \frac{2a}{a+1}\rho_a^n)/\rho_a^n$ converges to a constant depending on the parity of n, see (3.4), once again for $a = \frac{2}{3}$.

as $x \to \infty$, with $\widehat{c}_a$ an explicit constant given in (5.1).

Interestingly, the above estimates imply an asymmetry to the right of the random variable L_1, at the level of the density function. We notice that for $a \in (1/2, 1)$ (actually for $a \in (0,1)$), $\frac{2a-1}{2(1-a)} > \frac{2a^2-3a-1}{2(1-a^2)}$. Due to the construction of L_1, it is not surprising to have a heavier tail at $+\infty$ compared to $-\infty$. In fact, L_1 is the asymptotic variable when the process starts from a step $X_1 = +1$ a.s., with a memory parameter $p > 3/4$. The memory parameter being large, the random walk tends to repeat its previous steps, with a first step to the right. The elephant is therefore more likely to move to the right. Observe that this asymmetry is peculiar to the case $q = 1$. For $q \in (0,1)$ indeed, the random variable L_q has density $\varphi_q(x) = q\varphi(x) + (1-q)\varphi(-x)$ and the following corollary, which is an immediate consequence of Theorems 1.2 and 1.3, shows that this density has comparable tails at $\pm\infty$.

Corollary 1.4. *With the above notation, for $q \in (0,1)$, one has*

$$\varphi_q(x) \sim q\, c_a\, x^{\frac{2a-1}{2(1-a)}}\, e^{-(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}}} \qquad \text{and} \qquad \varphi_q(-x) \sim (1-q)\, c_a\, x^{\frac{2a-1}{2(1-a)}}\, e^{-(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}}}$$

as $x \to \infty$.

In the case $q = 1$, the difference between the two tails of the densities is specified in the next result, which is also a direct consequence of Theorems 1.2 and 1.3.

Corollary 1.5. *With the above notation, one has*

$$\frac{\varphi(x)}{\varphi(-x)} \sim \frac{4\,\Gamma\left(\frac{1-a}{1+a}\right)}{(a+1)^{\frac{2a}{1+a}}} \left(\frac{a}{\rho_a^{2-1/a}}\right)^{\frac{2a}{1-a^2}} x^{\frac{2a}{1-a^2}} \qquad \text{as } x \to \infty.$$

Our proofs of Theorems 1.2 and 1.3 are based on a combination of exact computations using special functions (such as hypergeometric functions, incomplete beta integrals and Mittag-Leffler functions) together with singularity analysis of generating functions (such as strong Tauberian theorems). Although the techniques for proving the two tails of the density are similar, the asymptotics on the half-negative line is significantly trickier. Let us present three main aspects of our methods.

The first step is to start from the moment calculation made in [20, Thm 1.4]

$$\mathbb{E}[L_1^n] = \frac{n!\, m_n}{\Gamma(1+an)}, \qquad \text{for all } n \geqslant 1, \tag{1.3}$$

and the non-linear recurrence relation for the sequence $\{m_n\}$ to be recalled in (3.5). We will deduce[1] an algebraic differential equation for the Stieltjes-like generating function

$$M(x) \,=\, \sum_{n\geqslant 0} m_n x^n \,=\, 1\,+\,x\,+\,\frac{a}{2a-1}x^2\,+\,\frac{a+1}{2(2a-1)}x^3\,+\cdots\,, \tag{1.4}$$

see (3.8). Surprisingly, despite its apparent complexity, the differential equation (3.8) admits a closed-form solution, which can be expressed in terms of a certain (incomplete beta) integrals, or equivalently in terms of some hypergeometric functions; see Proposition 3.2. In a sense, the explicit expression for $M(x)$ is an integrability property of the ERW.

In the second step, we prove that the series (1.4) has a first positive singularity at the point $\frac{1}{\rho_a}$, with ρ_a introduced in (1.2), and we obtain a series expansion at that point with the help of the exact expression given in the first step. Using the singularity analysis on generating functions developed in [16], we can deduce an asymptotic expansion of the moments, viewed as the coefficients of the power series (1.4). See Theorem 3.1 for the precise statement.

The third step aims at proving Theorems 1.2 and 1.3 on the asymptotic behavior of the density φ at $\pm\infty$. The general idea is to apply the strong Tauberian theorems of [15], where the fine properties of the exponential generating function

$$\Psi(r) \,=\, \mathbb{E}[e^{rL_1}] \,=\, \sum_{n\geqslant 0}\left(\frac{\mathbb{E}[L_1^n]}{n!}\right) r^n \tag{1.5}$$

can be transferred into asymptotic estimates of the density at $\pm\infty$. This is the most delicate part of the paper, which intrinsically requires an expansion of the integer moments up to the fifth order. The latter implies an expansion of $\Psi(r)$ in terms of generalized Mittag-Leffler functions and we can then apply the global estimates of Wright – see [29, 17] – on such functions in order to derive a sufficient control on Ψ and its derivatives and to perform accurately the inverse Laplace approximation method of [15]. For the reader's comfort, we also offer as an intermediate result in Proposition 3.7 a proof of the asymptotic behaviour at both infinities under the logarithmic scale, which is a simpler consequence of Kasahara's exponential Tauberian theorem.

2 Proof of Theorem 1.1

By definition, we have

$$n^{-a}S_n \xrightarrow{d} L_1$$

as $n\to\infty$. We first show that the random variable S_n, taking values in $\{2k-n,\, k=1,\ldots,n\}$, is unimodal for every $n\geqslant 1$ and all $p\in[0,1]$ (which covers cases where the ERW is not necessarily superdiffusive).

Setting $P(n,k)=\mathbb{P}[S_n=2k-n]$ and $Q(n,k)=(n-1)!P(n,k)$, it follows from the definition of the ERW that $\{Q(n,k),\, 1\leqslant k\leqslant n,\, n\geqslant 1\}$ is a triangular array defined recursively by $Q(1,1)=1$ and

$$Q(n+1,k) \,=\, (np-ak)Q(n,k)\,+\,(rn+a(k-1))Q(n,k-1) \tag{2.1}$$

[1] In general, if a power series (1.4) satisfies an algebraic differential equation, then its coefficients $\{m_n\}$ should satisfy a recurrence relation. On the other hand, the existence of a recurrence relation at the level of the coefficients does not necessarily imply that the associated power series satisfies a differential equation. In other words, the existence of a non-trivial differential equation for our power series $M(x)$ in (1.4) is a result in itself.

for all $n \geqslant 1$ and $1 \leqslant k \leqslant n+1$, where we have set $r = 1-p$ and, here and throughout, we make the convention $Q(n,0) = Q(n,n+1) = 0$. Let us briefly prove (2.1). The Markov property yields

$$\begin{aligned} P(n+1,k) \;\; &= \;\; \mathbb{P}[S_{n+1} = 2k-n-1 | S_n = 2k-n]\, P(n,k) \\ &\qquad\qquad + \mathbb{P}[S_{n+1} = 2k-n-1 | S_n = 2k-n-2]\, P(n,k-1). \end{aligned}$$

To conclude to (2.1), we multiply the above equation by $n!$, use the notation $Q(n,k)$ and the explicit transition probabilities of the ERW, namely

$$\left\{ \begin{array}{lcl} \mathbb{P}[S_{n+1} = 2k-n-1 | S_n = 2k-n] & = & \dfrac{1}{2}\left(1 - a\dfrac{2k-n}{n}\right), \\ \mathbb{P}[S_{n+1} = 2k-n-1 | S_n = 2k-n-2] & = & \dfrac{1}{2}\left(1 + a\dfrac{2k-n-2}{n}\right). \end{array} \right.$$

We need to show that for every $n \geqslant 1$ the finite sequence $\{Q(n,k),\, 1 \leqslant k \leqslant n\}$ is unimodal, that is – see e.g. Definition 4.1 in [13] – there exists a maximal index $k_n \in \{1, \ldots, n\}$ such that

$$\left\{ \begin{array}{ll} Q(n,k) \geqslant Q(n,k-1) & \text{for all } k \leqslant k_n, \\ Q(n,k) \leqslant Q(n,k-1) & \text{for all } k > k_n. \end{array} \right.$$

To do so, we use an induction on n starting from the unimodal sequence $\{Q(1,1) = 1\}$. Let $n \geqslant 1$ and suppose that the finite sequence $\{Q(n,k),\, 1 \leqslant k \leqslant n\}$ is unimodal. Decomposing

$$\begin{aligned} &Q(n+1,k+1) \,-\, Q(n+1,k) = \\ &\qquad (np - a(k+1))\big(Q(n,k+1) - Q(n,k)\big) \,+\, (nr + a(k-1))\big(Q(n,k) - Q(n,k-1)\big) \end{aligned}$$

implies that $Q(n+1,k+1) \geqslant Q(n+1,k)$ for all $k = 1, \ldots, k_n - 1$ by the induction hypothesis, since then one has $np - a(k+1) \geqslant np - ak_n \geqslant n(p-a) = n(1-p) \geqslant 0$. In particular, if $k_n = n$, the sequence $\{Q(n+1,k),\, 1 \leqslant k \leqslant n+1\}$ is unimodal with maximal index $k_{n+1} = n$ or $k_{n+1} = n+1$. In the case $k_n < n$, one necessarily has $Q(n,n) \leqslant Q(n,n-1)$ by the maximality of k_n, and we first show that this implies

$$p \;\leqslant\; \frac{n+1}{n+2}. \tag{2.2}$$

Indeed, if the contrary held, then we would also have $(k+2)p > k+1 \Leftrightarrow a(k+1) > kp$ for all $k = 1, \ldots, n$. Since on the other hand one has

$$Q(k+1,k+1) - Q(k+1,k) \;=\; (a(k+1) - kp)Q(k,k) + (kr + a(k-1))(Q(k,k) - Q(k,k-1))$$

for all $k = 1, \ldots, n$, an induction starting from $Q(1,0) = 0$ and $Q(1,1) = 1$ readily implies then $Q(k+1,k+1) > Q(k+1,k)$ for all $k = 1, \ldots, n$, a contradiction in the case $k = n-1$.

Therefore, since (2.2) implies $np \geqslant a(k+1)$ for all $k = 1, \ldots, n$, we also have $Q(n+1,k+1) \leqslant Q(n+1,k)$ for all $k = k_n+1, \ldots, n$. Putting everything together, we deduce that the sequence $\{Q(n+1,k),\, 1 \leqslant k \leqslant n+1\}$ is unimodal as required, with maximal index $k_{n+1} = k_n$ if $Q(n+1,k_n+1) < Q(n+1,k_n)$ and $k_{n+1} > k_n$ if $Q(n+1,k_n+1) \geqslant Q(n+1,k_n)$.

In order to complete the proof of the unimodality of L_1, we focus on the superdiffusive case $a \in (1/2,1)$ and consider the step function f_n defined by

$$f_n(x) \;=\; \frac{n^a P(n,k)}{2}$$

if $x \in (n^{-a}(2k-n-1), n^{-a}(2k-n+1)]$ for some $k \in \{1,\ldots,n\}$ and $f_n(x) = 0$ otherwise. It is clear from the above unimodality of S_n that $f_n(x)$ is a unimodal density function on $\mathbb{R}$. Moreover, it is easy to see by construction that the corresponding distribution function $F_n(x) = \int_{-\infty}^x f_n(y)\,dy$ is such that

$$\sup_{x\in\mathbb{R}} \left| F_n(x) \,-\, \mathbb{P}[n^{-a}S_n \leqslant x]\right| \,\leqslant\, \max\{P(n,k),\, k=1,\ldots,n\}.$$

Setting $m_n = \max\{Q(n,k),\, k=1,\ldots,n-1\}$ and

$$M_n \,=\, \max\{Q(n,k),\, k=1,\ldots,n\} \,=\, \max\{(n-1)!p^{n-1}, m_n\}$$

for all $n \geqslant 2$, we see by definition that $m_{n+1} \leqslant (n-a)M_n$ and $Q(n+1,n+1) = pnQ(n,n) \leqslant (n-a)M_n$ for n large enough. This shows that there exists $n_0 \geqslant 1$ such that $M_{n+1} \leqslant (n-a)M_n$ for all $n \geqslant n_0$, whence

$$\max\{P(n,k),\, k=1,\ldots,n\} \,=\, \frac{M_n}{(n-1)!} \,\leqslant\, \frac{M_{n_0}}{(1-a)_{n_0-1}}\left(\frac{(1-a)_{n-1}}{(n-1)!}\right) \to 0 \quad \text{as } n\to\infty.$$

Putting everything together shows that $F_n(x) \to \mathbb{P}[L_1 \leqslant x]$ for all $x \in \mathbb{R}$ and, by stability of unimodality under weak convergence – see e.g. Theorem 1.1 in [13], that L_1 is unimodal. ☐

Remark 2.1. (a) *If $\varphi'(0) > 0$, then the modes of L_1 are necessarily positive. Observe that for the Mittag-Leffler random variable with parameter $a > 1/2$, which is an approximation of L_1 from the point of view of integer moments and is also a strictly unimodal random variable, the mode is positive (whereas it is zero if $a \leqslant 1/2$). The known lower bound $M > \mathbb{E}[L_1] - \sqrt{3}\mathrm{Var}[L_1]$ where M is any mode of L_1 – see e.g. Lemma 1.7 in [13] implies that these modes are indeed positive for a close enough to 1.*

(b) *Simulations show that in the case $q \in (0,1)$, the random variable L_q is not unimodal in general. By* (1.1) *and Theorem 1.1 the density of L_q is the superposition of two unimodal functions and we conjecture that for every $a \in (1/2,1)$ there exists $q(a) \in [0,1/2]$ such that L_q is bimodal if $|q-1/2| < q(a)$ and unimodal if $|q-1/2| \geqslant q(a)$. This problem is believed to be challenging.*

We now show a refinement of the unimodality property for the ERW for certain values of the parameter a. Recall that a sequence $\{u_k,\, k=1,\ldots,n\}$ of positive numbers is log-concave if

$$u_k^2 \,\geqslant\, u_{k-1}u_{k+1}$$

for all $k = 1,\ldots,n$, where we have set $u_0 = u_{n+1} = 0$. It is well-known and easy to see that a log-concave sequence is always unimodal.

Proposition 2.2. *The sequence $\{\mathbb{P}[S_n = 2k-n],\, k=1,\ldots,n\}$ is log-concave for every $n \geqslant 1$ if and only if $a \in [-1, a_0]$ where $a_0 = 0.61803\ldots$ is the unique positive root of $a^3 + 4a^2 + 2a = 3$.*

Proof. With the above notation, we have to show that the sequence $\{Q(n,k),\, k=1,\ldots,n\}$ is log-concave for all $n \geqslant 1$ if and only if $a \in [-1,a_0]$. The sequences $\{Q(1,1)\}$ and $\{Q(2,1), Q(2,2)\}$ are always log-concave. We compute $Q(3,1) = (1-a)/2, Q(3,2) = (1-a)(2+a)/2, Q(3,3) = (1+a)^2/2$ and see that the log-concavity of $\{Q(3,1),Q(3,2),Q(3,3)\}$ amounts to

$$(1-a)(2+a)^2 \,\geqslant\, (1+a)^2 \iff 3 \,-\, 2a \,-\, 4a^2 \,-\, a^3 \,\geqslant\, 0 \iff a \in [-1,a_0],$$

which shows the only if part. We next suppose $a \in [-1,a_0]$ and show that $\{Q(n,k),\, k = 1,\ldots,n\}$ is log-concave for all $n \geqslant 1$ by an induction on n. From the above discussion, the property is true for $n = 1, 2, 3$ and we will show later that it is also true for $n = 4$. Assuming that $\{Q(n,k),\ k = 1, \ldots, n\}$ is log-concave for some $n \geqslant 4$, we first show

$$Q(n+1,n)^2 \;\geqslant\; Q(n+1,n+1)Q(n+1,n-1).$$

Setting $A_n = Q(n+1,n)^2 \,-\, Q(n+1,n+1)Q(n+1,n-1)$ and using (2.1) we compute, after some simplifications,

$$\begin{aligned}
4A_n &= n^2(1-a)^2Q(n,n)^2 \,+\, n(n(1-a^2) - 6a + 2a^2)Q(n,n)Q(n,n-1) \\
&+ \; n(1+a)(n(1+a) - 4a))(Q(n,n-1)^2 \,-\, Q(n,n)Q(n,n-2)) \,+\, 4a^2Q(n,n-1)^2 \\
&\geqslant n^2(1-a)^2Q(n,n)^2 \,+\, n(4 - 6a - 2a^2)Q(n,n)Q(n,n-1) \,+\, 4a^2Q(n,n-1)^2 \\
&\geqslant n^2(1-a)^2Q(n,n)^2 \,-\, 4na(1-a)Q(n,n)Q(n,n-1) \,+\, 4a^2Q(n,n-1)^2 \\
&= (n(1-a)Q(n,n) \,-\, 2aQ(n,n-1))^2 \,\geqslant\, 0,
\end{aligned}$$

where in the first inequality we have used the induction hypothesis and $n \geqslant 4$, and in the second inequality we have used $a \leqslant a_0 \leqslant 2/3$ which implies $4 - 6a - 2a^2 + 4a(1-a) = 2(1+a)(2-3a) \geqslant 0$. We next prove

$$Q(n+1,k)^2 \;\geqslant\; Q(n+1,k+1)Q(n+1,k-1)$$

for all $k = 2, \ldots, n-1$. Setting $B_{k,n} = Q(n+1,k)^2 - Q(n+1,k+1)Q(n+1,k-1)$ and using (2.1) again we compute

$$\begin{aligned}
B_{k,n} &= (np - a(k+1))(np - a(k-1))B_{k,n-1} \,+\, (nr + ak)(nr + a(k-2)B_{k-1,n-1} \\
&+\; a^2\left(Q(n,k) - Q(n,k-1)\right)^2 - (np - a(k+1))(nr + a(k-2))Q(n,k-2)Q(n,k+1) \\
&+\; \left(2a^2 + 2(np - ak)(nr + a(k-1)) - (nr + ak)(np - a(k-1))\right)Q(n,k-1)Q(n,k) \\
&\geqslant (np - a(k+1))(nr + a(k-2))\left(Q(n,k-1)Q(n,k) - Q(n,k-2)Q(n,k+1)\right) \\
&\geqslant 0,
\end{aligned}$$

where the second equality follows from the induction hypothesis. To complete the proof, it remains to show that $\{Q(n,k),\ k = 1, \ldots, n\}$ is log-concave for all $a \in [-1, a_0]$ and $n = 4$. The above discussion and the case $n = 3$ already implies $Q(4,2)^2 \geqslant Q(4,1)Q(4,3)$ and we hence just need to check that $A_3 \geqslant 0$, with the above notation, which amounts after some simplifications to

$$P(a) \,=\, 54 \,+\, 36a \,-83a^2 \,-\, 123a^3 \,-\, 63a^4 \,-\, 13a^5 \,\geqslant\, 0.$$

The polynomial $P(a)$ is easily seen to decrease on $[1/2, 1]$ with $P(1/2) > 0$ and $P(1) < 0$ and has hence a unique root $a_1 = 0.63606\ldots > a_0$, so that $P(a) \geqslant 0$ for all $a \in [1/2, a_0]$. Moreover, we have

$$\begin{aligned}
4A_3 &\geqslant 9(1-a)^2Q(3,3)^2 \,+\, 3(3 - 6a - a^2)Q(3,3)Q(3,2) \,+\, 4a^2Q(3,2)^2 \\
&\geqslant (3(1-a)Q(3,3) \,-\, 2aQ(3,2))^2 \,\geqslant\, 0
\end{aligned}$$

as well for all $a \in [-1, 1/2]$, since then $3 - 6a - a^2 + 4a(1-a) = (1+a)(3-5a) \geqslant 0$. $\square$

Remark 2.3. *One can show that there exists an increasing sequence $\{a_m, m \geqslant 0\}$ such that*

$$\{Q(n,k),\ k = 1, \ldots, n\} \text{ is log-concave for all } n \geqslant m+3 \iff a \,\leqslant\, a_m$$

for all $m \geqslant 0$*, and that* x_m *is the unique root on* $(1/2, 1)$ *of a certain real polynomial* P_m *whose degree is smaller than* $2m+3$*. One has*

$$\begin{aligned}
P_0 &= 3 - 2X - 4X^2 - X^3, \\
P_1 &= 54 + 36X - 83X^2 - 123X^3 - 63X^4 - 13X^5, \\
P_2 &= 360 + 834X + 247X^2 - 1259X^3 - 2009X^4 - 1423X^5 - 514X^6 - 76X^7, \\
P_3 &= 11250 + 8325X - 13781X^2 - 24282X^3 - 13396X^4 - 1024X^5 \\
&\qquad\qquad + 2518X^6 + 1361X^7 + 229X^8,
\end{aligned}$$

with $a_0 \simeq 0.61803, a_1 \simeq 0.63606, a_2 \simeq 0.67060$ *and* $a_3 \simeq 0.68408$*. We conjecture that*

$$\lim_{m\to\infty} a_m \;=\; 1.$$

However, this problem seems challenging because of the very complicated character of the underlying polynomials.

The following is an interesting consequence of Proposition 2.2.

Corollary 2.4. *The density of* L_1 *is log-concave on* $\mathbb{R}$ *for all* $a \in (1/2, a_0]$*.*

Proof. The argument is analogous to that of the end of the proof of Theorem 1.1 except that we consider here the piecewise affine density g_n defined for all $n \geqslant 2$ by

$$g_n(x) \;=\; \frac{n^a}{2-(P(1,n)+P(n,n))}\left(P(k,n) + \frac{n^a(P(k+1,n)-P(k,n))(x+n-2k)}{2}\right)$$

if $x \in (n^{-a}(2k-n), n^{-a}(2k-n+2)]$ for some $k \in \{1,\ldots,n-1\}$ and $g_n(x) = 0$ otherwise. The log-concavity of the sequence $\{P(n,k),\ k = 1,\ldots,n\}$ readily implies the log-concavity of the function g_n on $\mathbb{R}$ for all $a \in (1/2, a_0]$, and it follows easily from Scheffé's lemma and the end of the proof of Theorem 1.1 that the real random variable with density g_n converges weakly to L_1 as $n \to \infty$. Since log-concavity is a stable property under weak convergence – see e.g. Theorem 2.10 in [13], this shows altogether that L_1 has a log-concave density for all $a \in (1/2, a_0]$. $\square$

Remark 2.5. *A positive answer to the open problem stated in Remark 2.3 would imply by the same approximation argument that* L_1 *has a log-concave density on* $\mathbb{R}$ *for all* $a \in (1/2, 1)$*. Observe that this property is in accordance with the superexponential behaviour of the density at* $\pm\infty$ *stated in Theorems 1.2 and 1.3.*

3 Moment estimates

In this section we establish sharp estimates for the positive integer moments of L_1, which play a crucial role in the proof of Theorems 1.2 and 1.3, and have an independent interest. The principal estimate is given in (3.3) below. However, the proof of Theorem 1.2 requires an expansion up to the third order whereas that of Theorem 1.3 needs an expansion up to the fifth order, which is given in (3.4) with the notation

$$\kappa_a \;=\; \frac{\rho_a^{\frac{2}{a+1}}}{4}\left(\frac{a+1}{a}\right)^{\frac{2a}{a+1}} \qquad \text{and} \qquad \delta_a \;=\; \frac{1-a}{1+a}\cdot \tag{3.1}$$

Our main tool to get the moment estimates is the singularity analysis of [16], where expansions are given in terms of Pochhammer symbols which we recall to be defined by $(x)_0 = 1$ and $(x)_n = x(x+1)\cdots(x+n-1)$, for all $n \geqslant 1$ and $x \in \mathbb{R}$. This can be easily transformed into an expansion in descending powers of n thanks to the asymptotics

$$\frac{(x)_n}{n!} \;=\; \frac{\Gamma(x+n)}{\Gamma(x)\Gamma(1+n)} \;=\; \frac{n^{x-1}}{\Gamma(x)}\left(1 \,+\, \frac{x(x-1)}{2n} \,+\, \mathrm{O}\left(n^{-2}\right)\right), \tag{3.2}$$

which is valid for all $x > 0$ – see [23] for a more complete version. The asymptotic expansion with Pochhammer symbols will also be convenient along the proof of Theorems 1.2 and 1.3, thanks to the direct connection with generalized Mittag-Leffler functions and their behaviour at infinity.

Theorem 3.1. *One has*

$$\mathbb{E}[L_1^n] \sim \frac{2a\,\rho_a^n\,n!}{(a+1)\,\Gamma(1+an)} \qquad \text{as } n \to \infty. \tag{3.3}$$

More precisely, one has the asymptotic expansion

$$\begin{aligned}\mathbb{E}[L_1^n] \;=\; \frac{2a\,\rho_a^n\,n!}{(a+1)\,\Gamma(1+an)} \Bigg(1 \,+\, \kappa_a\,\frac{(\delta_a)_n}{n!}\left((-1)^n \,+\, \frac{a-1}{3a+1}\right) \\ +\,\frac{(2\delta_a)_n}{(n+1)!}\left((-1)^n\kappa_1^- + \kappa_1^+\right) \,+\, \frac{(\delta_a)_n}{(n+1)!}\left((-1)^n\kappa_2^- + \kappa_2^+\right) \,+\, \mathrm{O}\left(n^{-2}\right)\Bigg)\end{aligned} \tag{3.4}$$

as $n \to \infty$, where ρ_a is introduced in (1.2)*, κ_a and δ_a in* (3.1)*, $\kappa_1^-, \kappa_1^+, \kappa_2^-$ and κ_2^+ are computable real constants depending on a.*

The proof of this theorem relies on the representation (1.3), where the sequence $\{m_n\}$ is defined by $m_0 = m_1 = 1$ and

$$m_n \;=\; \frac{1}{na - c_n}\sum_{i=1}^{n-1} c_i m_i m_{n-i}, \tag{3.5}$$

for every $n \geqslant 2$, where $c_i = 1$ for even i and $c_i = a$ for odd i. By Lemma 2.10 and Corollary 2.14 in [20], for every $a \in \left(\frac{1}{2}, 1\right)$ there exists $C_a > 0$ such that $0 < m_n \leqslant C_a^n$ for all $n \geqslant 0$, which shows that the generating function (1.4) has a positive radius of convergence. We will first give an exact expression of $M(x)$, showing that this radius is actually $1/\rho_a$, with ρ_a introduced in (1.2). Set

$$F(x) \;=\; \frac{1}{a}\int_x^\infty \frac{du}{u^{1+1/a}\sqrt{1+u^2}} \tag{3.6}$$

and consider the function

$$G(x) \;=\; F^{-1}(x^{-1/a} - \rho_a^{1/a}) \tag{3.7}$$

for every $x \in (0, 1/\rho_a)$, where F^{-1} denotes the compositional inverse of the decreasing function F on $(0, \infty)$.

Proposition 3.2. *For every $x \in (0, 1/\rho_a)$, one has*

$$M(x) \;=\; \left(\frac{G(x)}{x}\right)^{\frac{1}{a}}\left(G(x) + \sqrt{1 + G(x)^2}\right),$$

with G defined by (3.7).

The next paragraph is devoted to the proof of this proposition. Then, in Section 3.2, we will analyze precisely the behaviour of $M(x)$ as $x \to 1/\rho_a$ and prove Theorem 3.1.

3.1 Proof of Proposition 3.2

Multiplying the recurrence relation (3.5) by $(na - c_n)$ we obtain that for all $n \geqslant 0$,

$$anm_n = \sum_{i=1}^{n} c_i m_i m_{n-i} - m_n.$$

Multiplying by x^n and taking the generating functions yields first the following ODE for the series $M(x)$ in (1.4):

$$M(x) \,+\, axM'(x) \,-\, pM(x)^2 \,-\, (1-p)M(x)M(-x) \,=\, 0, \tag{3.8}$$

which appears as a mix of an ordinary (algebraic) differential equation and a functional equation, because of the term $M(-x)$. Separating $M(x) = A(x) + B(x)$ with

$$A(x) \,=\, \sum_{n\geqslant 0} m_{2n}x^{2n} \qquad \text{and} \qquad B(x) \,=\, \sum_{n\geqslant 0} m_{2n+1}x^{2n+1},$$

the above equation can be rewritten as

$$A \,+\, B \,+\, axA' \,+\, axB' \,-\, p(A^2+2AB+B^2) \,-\, (1-p)(A^2-B^2) \,=\, 0$$

and isolating the even and odd parts, we obtain the differential system

$$\begin{cases} A \,+\, axA' \,-\, p(A^2+B^2) \,-\, (1-p)(A^2-B^2) \,=\, 0, \\ B \,+\, axB' \,-\, 2pAB \,=\, 0. \end{cases} \tag{3.9}$$

Using the second equation we express A in terms of B and B', and then plugging this identity in the first equation, we obtain after cleaning denominators and simplification an autonomous, classical ODE on $B(x)$, which reads

$$a(a+1)x^2BB'' - a(a+2)x^2B'^2 + x((a+1)^2-2)BB' - (a+1)^2B^4 + B^2 \,=\, 0. \tag{3.10}$$

The following lemma shows that $B(x)$ is also the solution to an implicit equation, involving the function F in (3.6). It is worth noticing that Lemma 3.3 is the first place where the non-trivial quantity ρ_a appears.

Lemma 3.3. *For every $x \in (0, 1/\rho_a)$, the function $B(x)$ is the unique solution of*

$$x^{1/a}F\left((x^{1/a}B(x))^{a/(a+1)}\right) \,+\, (\rho_a x)^{1/a} \,=\, 1. \tag{3.11}$$

In particular, one has $B(x) \to \infty$ as $x \to 1/\rho_a$ and the convergence radius of the series defining $B(x)$ is $1/\rho_a$.

Proof. We first remark that the solution of (3.11) exists and is uniquely defined as

$$B(x) \,=\, x^{-1/a}\left(F^{-1}(x^{-1/a} - \rho_a^{1/a})\right)^{\frac{a+1}{a}}, \tag{3.12}$$

which is a smooth function on $(0, 1/\rho_a)$. Moreover, since

$$\begin{aligned} x^{1/a}F\left((x^{1/a}B(x))^{a/(a+1)}\right) \quad &= \quad \frac{x^{1/a}}{a}\int_{(x^{1/a}B(x))^{a/(a+1)}}^{\infty} \frac{\mathrm{d}u}{u^{1+1/a}}\left(\frac{1}{\sqrt{1+u^2}} - 1\right) \\ &\qquad\qquad + \left(\frac{x}{B(x)}\right)^{1/(a+1)} \end{aligned}$$

and since for any $a \in (1/2, 1)$, the integral on the right-hand side is bounded as a function of x, we obtain $\lim_{x\to 0}\frac{x}{B(x)} = 1$, whence B is right-differentiable at zero with $B(0) = 0$ and $B'(0) = 1$.

We now prove that B satisfies (3.10). Dividing (3.11) by $x^{1/a}$ and differentiating with respect to x we obtain, after a few elementary computations,

$$\frac{1}{a+1}\left(1 + ax\frac{B'}{B}\right) = \left(\frac{B}{x}\right)^{\frac{1}{a+1}}\sqrt{1 + x^{\frac{2}{a+1}}B^{\frac{2a}{a+1}}}.$$

Taking the square of the previous identity and simplifying, we get

$$\left(\frac{B}{x}\right)^{\frac{2}{a+1}} = -B^2 + \frac{1}{(a+1)^2}\left(1 + ax\frac{B'}{B}\right)^2. \tag{3.13}$$

Differentiating (3.13) with respect to x and multiplying by $\frac{B}{x}$, we obtain

$$\frac{2}{a+1}\left(\frac{B}{x}\right)'\left(\frac{B}{x}\right)^{\frac{2}{a+1}} = \frac{B}{x}\frac{\mathrm{d}}{\mathrm{d}x}\left(-B^2 + \frac{1}{(a+1)^2}\left(1 + ax\frac{B'}{B}\right)^2\right). \tag{3.14}$$

Using both (3.13) and (3.14), we can eliminate the term $(\frac{B}{x})^{\frac{2}{a+1}}$ (which contains a non-integer power of B), and obtain a classical ODE on B. After a few simplifications, we exactly obtain (3.10).

We now show that any solution B of (3.10) with $B(0)=0$ and $B'(0)=1$ must satisfy (3.11), which will complete the proof. Observe that by (3.10), we necessarily have $B''(0)=0$. Moreover, the previous discussion shows that for any $\alpha>0$, the function

$$x^{-1/a} \;-\; \frac{1}{a}\int_{\left(x^{1/a}B(x)\right)^{a/(a+1)}}^{\alpha} \frac{\mathrm{d}u}{u^{1+1/a}\sqrt{1+u^2}}$$

is a constant c_α. It remains to compute this constant. We observe that

$$c_\alpha \;=\; \frac{1}{a}\int_{\left(x^{1/a}B(x)\right)^{a/(a+1)}}^{\alpha}\left(1-\frac{1}{\sqrt{1+u^2}}\right)\frac{\mathrm{d}u}{u^{1+1/a}} \;+\; x^{-1/a}\left(1-\left(\frac{x}{B(x)}\right)^{\frac{1}{a+1}}\right) \;+\; \alpha^{-1/a}.$$

Since $2-\frac{1}{a}>0$, taking the limit when $x\to 0$, we deduce

$$c_\alpha \;=\; \frac{1}{a}\int_0^{\alpha}\left(1-\frac{1}{\sqrt{1+u^2}}\right)\frac{\mathrm{d}u}{u^{1+1/a}} \;+\; \alpha^{-1/a} \;\to\; \frac{1}{a}\int_0^{\infty}\left(1-\frac{1}{\sqrt{1+u^2}}\right)\frac{\mathrm{d}u}{u^{1+1/a}}$$

as $\alpha\to\infty$. We finally compute

$$\begin{aligned}
\frac{1}{a}\int_0^{\infty}\left(1-\frac{1}{\sqrt{1+u^2}}\right)\frac{\mathrm{d}u}{u^{1+1/a}} \quad &= \quad \frac{1}{a}\int_0^{\infty}\frac{\cosh v - 1}{(\sinh v)^{1+1/a}}\,\mathrm{d}v \\
&= \quad \frac{2^{-1/a}}{a}\int_0^{\infty}\frac{\mathrm{d}v}{\left(\sinh\frac{v}{2}\right)^{1/a-1}\left(\cosh\frac{v}{2}\right)^{1+1/a}} \\
&= \quad \frac{2^{-1/a}}{a}\int_0^{\infty}\frac{\mathrm{d}x}{x^{1/(2a)}(1+x)^{1+1/(2a)}} \\
&= \quad \frac{2^{-1/a}\Gamma\left(1+\frac{1}{a}\right)}{\Gamma\left(1+\frac{1}{2a}\right)}\,\Gamma\left(1-\frac{1}{2a}\right) \\
&= \quad \frac{\mathrm{B}\left(\frac{1}{2}+\frac{1}{2a},1-\frac{1}{2a}\right)}{2} \;=\; \rho_a^{1/a}
\end{aligned}$$

where we have used the changes of variable $u=\sinh v$ and $x=\left(\sinh\frac{v}{2}\right)^2$ and the duplication formula for the Gamma function. Putting everything together shows that any solution B of (3.10) with $B(0)=0$ and $B'(0)=1$ satisfies (3.11), as required. □

We can now finish the proof of Proposition 3.2. The above Lemma 3.3 and the second equation in (3.9) show after some light algebraic computations that

$$B(x) \;=\; x\left(\frac{G(x)}{x}\right)^{\frac{a+1}{a}} \qquad \text{and} \qquad A(x) \;=\; \left(\frac{G(x)}{x}\right)^{\frac{1}{a}}\sqrt{1+G(x)^2} \tag{3.15}$$

for every $x\in(0,1/\rho_a)$, whence the required expression for the generating function $M(x)$.

□

3.2 Proof of Theorem 3.1

We will use the convergent series representations

$$x^{-1/a} - \rho_a^{1/a} = \frac{\rho_a^{1/a}(1-x\rho_a)}{a} \sum_{n\geqslant 0} \frac{(1+1/a)_n}{(n+1)!}\,(1-x\rho_a)^n \tag{3.16}$$

for every $x \in (0, 1/\rho_a)$, and

$$F(z) = z^{-(1+1/a)} \sum_{n\geqslant 0} \frac{(-1)^n (1/2)_n}{(1+a+2an)n!}\, z^{-2n} \tag{3.17}$$

for every $z > 1$. The first one is a simple consequence of $x^{-1/a} = \rho_a^{1/a}\big(1-(1-x\rho_a)\big)^{-1/a}$ and the binomial theorem, whereas the second one follows from the change of variable

$$F(z) = \frac{1}{2a}\int_0^{1/z^2} v^{\frac{1-a}{2a}} \frac{dv}{\sqrt{1+v^2}} = \frac{1}{2a}\sum_{n\geqslant 0}\frac{(-1)^n(1/2)_n}{n!}\left(\int_0^{1/z^2} v^{\frac{1-a}{2a}+n}\,dv\right),$$

see (3.6), where in the second equality we have applied the binomial theorem to $(1+v^2)^{-1/2}$ and the involved series is easily seen to be absolutely convergent for all $z > 1$, so that the switching between the sum and the integral is justified. The Lagrange inversion formula – see e.g. Appendix E in [1] – implies then after some computations the expansion

$$F^{-1}(y) \sim \left(\frac{1}{(a+1)y}\right)^{\frac{a}{a+1}} \left(1 - \sum_{j=1}^{\infty} c_j\, y^{\frac{2ja}{a+1}}\right) \qquad \text{as } y \to 0,$$

with

$$c_1 = \frac{a(a+1)^{\frac{2a}{a+1}}}{2(3a+1)}.$$

The further constants c_p can be computed explicitly from (3.17) and, here and throughout, we have used the usual notation $\sim$ for asymptotic expansions – see e.g. Appendix C of [1]. This entails

$$F^{-1}(y)^{\frac{a+1}{a}} \sim \frac{1}{(a+1)y}\left(1 - \sum_{j=1}^{\infty} \widetilde{c}_j\, y^{\frac{2ja}{a+1}}\right) \qquad \text{as } y \to 0, \tag{3.18}$$

with

$$\widetilde{c}_1 = \frac{(a+1)^{\frac{3a+1}{a+1}}}{2(3a+1)}$$

and where the $\widetilde{c}_p$ can be computed recursively from the c_p. In order to obtain our moment estimates, we will combine Proposition 3.2, Equations (3.12), (3.16) and (3.18) and the classical singularity analysis of [16]. First, observe that (3.17) can be expressed as a hypergeometric series,

$$F(z) = \frac{z^{-(1+1/a)}}{a+1}\, {}_2F_1\!\left[\begin{matrix} 1/2 \quad 1/2+1/(2a) \\ 3/2+1/(2a)\end{matrix}; -1/z^2\right], \tag{3.19}$$

which shows by Euler's integral formula – see e.g. Theorem 2.2.1 in [1] – that the function F is holomorphic on $\mathbb{C} \setminus \big((-\infty, 0] \cup \{iu, u \in [-1,1]\}\big)$. It will actually turn out from the alternative representation (3.21) below that this function is holomorphic on the whole cut plane $\mathbb{C} \setminus (-\infty, 0]$. This implies that there exists $\varepsilon > 0$ such that the function $z \mapsto F^{-1}(z)^{\frac{a+1}{a}}$ appearing in (3.18) is holomorphic on the cut open disk $\{|z| < \varepsilon\} \setminus (-\infty, 0]$. Writing again

$$z^{-1/a} - \rho_a^{1/a} = \rho_a^{1/a}\left(\big(1-(1-z\rho_a)\big)^{-1/a} - 1\right)$$

and applying Proposition 3.2 show that there exists $\eta > 0$ such that the function $z \mapsto M(z)$ is holomorphic on the indented open disk $\Delta_{a,\eta} = \{|z - 1/\rho_a| < \eta\} \cap \{|\arg(z - 1/\rho_a)| > (1-a)\pi\}$. Since $a > 1/2$, we are in position to apply [16, Thm 1] to the generating function M in (1.4) around the critical point $z = 1/\rho_a$. Plugging (3.16) in (3.18) and applying Proposition 3.2 imply

$$M(z) - \frac{2a}{(a+1)(1-z\rho_a)} = \mathrm{O}\big((1-z\rho_a)^{-\delta_a}\big) \qquad \text{as } z \to 1/\rho_a \text{ inside } \Delta_{a,\eta}.$$

Therefore, using $\delta_a < 1$ and applying Theorem 1 in [16], we obtain

$$m_n \sim \frac{2a\,\rho_a^n}{a+1}$$

which gives the principal estimate (3.3). See Figure 2 for an illustration of the above asymptotic result.

In order to get the higher order asymptotics (3.4), we have to handle the odd and even moments separately. We shall begin with the odd moments and consider the renormalized sequence $\mu_n = \rho_a^{-2n-1} m_{2n+1}$, whose generating function reads

$$\sum_{n\geqslant 0} \mu_n u^n = \frac{B(x)}{\rho_a\, x} = \frac{1}{\rho_a}\left(\frac{G(x)}{x}\right)^{1+1/a} = \rho_a^{1/a}\left(\sum_{n\geqslant 0}\frac{(1/2+1/(2a))_n}{n!}\,(1-u)^n\right) G(x)^{1+1/a},$$

where we have set $u = x^2\rho_a^2$, the second equality is given by (3.15), and the third equality follows from the binomial theorem applied to $x^{-1-1/a} = \rho_a^{1+1/a}\big((1-(1-u)\big)^{-(1/2+1/(2a))}$. Using (3.18) and plugging in the series representation

$$\begin{aligned} y = x^{-1/a} - \rho_a^{1/a} &= \frac{\rho_a^{1/a}(1-u)}{2a}\sum_{n\geqslant 0}\frac{(1+1/(2a))_n}{(n+1)!}\,(1-u)^n \\ &= \frac{\rho_a^{1/a}(1-u)}{2a}\left(1 + \frac{2a+1}{4a}(1-u) + \mathrm{O}(1-u)^2\right) \end{aligned} \tag{3.20}$$

which is a variation on (3.16), we deduce the expansion

$$\begin{aligned} \sum_{n\geqslant 0}\mu_n u^n \;&\sim\; \frac{2a}{(a+1)(1-u)} \times \frac{\left(\sum_{n\geqslant 0}\frac{(1/2+1/(2a))_n}{n!}\,(1-u)^n\right)\left(1 - \sum_{j=1}^{\infty}\widetilde{c}_j\, y^{\frac{2ja}{a+1}}\right)}{\sum_{n\geqslant 0}\frac{(1+1/(2a))_n}{(n+1)!}\,(1-u)^n} \\ &\sim\; \frac{2a}{(a+1)(1-u)}\left(1 + \frac{1-u}{4a} + \mathrm{O}(1-u)^2\right)\left(1 - \sum_{j=1}^{\infty}\widetilde{c}_j\, y^{\frac{2ja}{a+1}}\right) \end{aligned}$$

as $u \to 1$, which reads

$$\frac{2a}{a+1}\left(\frac{1}{1-u} - \frac{\widehat{c}_1}{(1-u)^{\delta_a}} + \frac{1}{4a} + \widehat{c}_2(1-u)^{1-2\delta_a} + \widehat{c}_3(1-u)^{1-\delta_a} + \mathrm{O}(1-u)\right)$$

with

$$\widehat{c}_1 = \frac{\widetilde{c}_1\,\rho_a^{\frac{2}{a+1}}}{(2a)^{\frac{2a}{a+1}}} = 2^{\delta_a}\kappa_a\left(\frac{a+1}{3a+1}\right)$$

and $\widehat{c}_2, \widehat{c}_3$ are computable constants. Applying Corollary 3 in [16], we deduce

$$\begin{aligned}\mu_n &= \frac{2a}{a+1}\left(1 - \frac{\widehat{c}_1\,(\delta_a)_n}{n!} + \frac{\widehat{c}_2(2\delta_a-1)_n}{n!} + \frac{\widehat{c}_3(\delta_a-1)_n}{n!} \,+\, \mathrm{O}\left(n^{-2}\right)\right)\\ &= \frac{2a}{a+1}\left(1 - \frac{\bar{c}_1(\delta_a)_{2n+1}}{(2n+1)!} + \frac{\bar{c}_2\,(2\delta_a)_{2n+1}}{(2n+2)!} + \frac{\bar{c}_3\,(\delta_a)_{2n+1}}{(2n+2)!} \,+\, \mathrm{O}\big(2n+1\big)^{-2}\right)\end{aligned}$$

with

$$\bar{c}_1 \;=\; 2\kappa_a\left(\frac{a+1}{3a+1}\right)$$

and $\bar{c}_2, \bar{c}_3$ are computable constants and where in the second equality we have used (3.2), which implies

$$\frac{(\delta_a)_n}{n!} \;=\; 2^{\frac{2a}{a+1}}\frac{(\delta_a)_{2n+1}}{(2n+1)!}\left(1 \,+\, \frac{2a(3+a)}{(a+1)^2 n} \,+\, \mathrm{O}(n^{-2})\right).$$

We now proceed to the even moments and consider the sequence $\nu_n = \rho_a^{-2n} m_{2n}$, whose generating function reads, similarly as above,

$$\sum_{n\geqslant 0}\nu_n u^n \;=\; A(x) \;=\; \left(\frac{G(x)}{x}\right)^{1/a}\sqrt{1+G(x)^2} \;=\; B(x)\left(\sum_{n\geqslant 0}\frac{(-1/2)_n(-1)^n}{n!}\,G(x)^{-2n}\right)$$

with the same notation $u = \rho_a^2 x^2$. This implies the expansion

$$\begin{aligned}\sum_{n\geqslant 0}\nu_n u^n &= \sqrt{u}\left(\sum_{n\geqslant 0}\mu_n u^n\right)\left(\sum_{n\geqslant 0}\frac{(-1/2)_n(-1)^n}{n!}\,G(x)^{-2n}\right)\\ &= \left(1 \,-\, \frac{1-u}{2}\right)\left(\sum_{n\geqslant 0}\mu_n u^n\right)\left(1 \,+\, \frac{1}{2G(x)^2} \,-\, \frac{1}{8G(x)^4}\right) \,+\, \mathrm{O}(1-u)^2\end{aligned}$$

as $u \to 1$. Using

$$\frac{1}{G(x)^2} \;=\; ((a+1)y)^{\frac{2a}{a+1}}\left(1 \,+\, \sum_{j\geqslant 1} c_j\, y^{\frac{2ja}{a+1}}\right)^{-2}$$

with the expansion (3.20), and the previous estimates on the odd moments we deduce, after some simplifications,

$$\sum_{n\geqslant 0}\nu_n u^n \sim$$
$$\frac{2a}{a+1}\left(\frac{1}{1-u} \,+\, \frac{\widehat{c}_4}{(1-u)^{\delta_a}} \,+\, \frac{1-2a}{4a} \,+\, \widehat{c}_5(1-u)^{1-2\delta_a} \,+\, \widehat{c}_6(1-u)^{1-\delta_a} \,+\, \mathrm{O}(1-u)\right)$$

as $u \to 1$, with

$$\widehat{c}_4 \;=\; \left(\frac{a\,\rho_a^{\frac{2}{a+1}}}{3a+1}\right)\left(\frac{a+1}{2a}\right)^{\frac{2a}{a+1}} \;=\; 2^{1+\delta_a}\kappa_a\left(\frac{a}{3a+1}\right)$$

and $\widehat{c}_5, \widehat{c}_6$ are computable constants. Applying again Corollary 3 in [16], we obtain

$$\begin{aligned}\nu_n &= \frac{2a}{a+1}\left(1 \,+\, \frac{\widehat{c}_4\,(\delta_a)_n}{n!} \,+\, \frac{\widehat{c}_5\,(2\delta_a-1)_n}{n!} \,+\, \frac{\widehat{c}_6\,(\delta_a-1)_n}{n!} \,+\, \mathrm{O}\left(n^{-2}\right)\right)\\ &= \frac{2a}{a+1}\left(1 \,+\, \frac{\bar{c}_4\,(\delta_a)_{2n}}{(2n)!} \,+\, \frac{\bar{c}_5\,(2\delta_a)_{2n}}{(2n+1)!} \,+\, \frac{\bar{c}_6\,(\delta_a)_{2n}}{(2n+1)!} \,+\, \mathrm{O}\left((2n)^{-2}\right)\right)\end{aligned}$$

with

$$\bar{c}_4 \;=\; \frac{4a\,\kappa_a}{3a+1}$$

and $\bar{c}_5, \bar{c}_6$ are computable constants. Putting together the expansions of odd and even moments completes the proof. □

Remark 3.4. *Applying the transformation (2.3.12) in [1] to the hypergeometric expression* (3.19) *gives the series representation*

$$F(z) \;=\; -\rho_a^{1/a} \;+\; z^{-1/a}\,{}_2F_1\left[\begin{matrix}1/2 \quad -1/(2a) \\ 1-1/(2a)\end{matrix}; -z^2\right], \tag{3.21}$$

which defines an analytic function on the cut disk $\{|z| \leqslant 1\} \setminus (-\infty, 0]$. *By* (3.12)*, this shows that the function* $C(x) = B(x)^{\frac{a}{a+1}} x^{\frac{1}{a+1}}$ *is the solution to the implicit equation*

$$C(x)\left({}_2F_1\left[\begin{matrix}1/2 \quad -1/(2a) \\ 1-1/(2a)\end{matrix}; -C(x)^2\right]\right)^{-a} \;=\; x,$$

which can be inverted by the Lagrange formula and we retrieve

$$B(x) \;=\; x \;+\; \frac{a+1}{2(2a-1)}x^3 \;+\; \cdots$$

Observe that (3.21) *also gives a direct proof of the identity*

$$\rho_a^{1/a} \;=\; \frac{1}{a}\int_0^\infty \left(1 - \frac{1}{\sqrt{1+u^2}}\right)\frac{\mathrm{d}u}{u^{1+1/a}} \;=\; \lim_{x\downarrow 0}\left(x^{-1/a} \;-\; F(x)\right),$$

which was used at the end of Lemma 3.3.

3.3 Elementary properties of the exponential growth

We have the following property of the exponential rate of growth of the sequence $\{m_n\}$. See Figure 2.

Proposition 3.5. *The function* $a \mapsto \rho_a$ *is decreasing and convex on* $(1/2, 1)$ *from* ∞ *to 1 with*

$$\rho_a \;\sim\; \frac{1}{2\sqrt{a-1/2}} \quad \text{as } a \to 1/2 \qquad \text{and} \qquad \rho_a - 1 \;\sim\; (1-a)\log 2 \quad \text{as } a \to 1.$$

Proof. We start with the decreasing property. Setting $b = 1/(2a) \in (1/2, 1)$, we need to show that function

$$f(b) \;=\; \left(\frac{\Gamma(1/2+b)\Gamma(1-b)}{\Gamma(1/2)\Gamma(1)}\right)^{\frac{1}{2b}} \;=\; \left(\prod_{n\geqslant 0}\frac{(1/2+n)(1+n)}{(1/2+b+n)(1-b+n)}\right)^{\frac{1}{2b}}$$

increases on $(1/2, 1)$, where in the second equality we have used the classical product representation of the Gamma function – see e.g. Theorem 1.1.2 in [1]. Taking the logarithm, we are reduced to show that the function $b \mapsto b^{-1}g_n(b)$ increases on $(1/2, 1)$ for each $n \geqslant 0$, with

$$g_n(b) \;=\; \log(1/2+n) \;+\; \log(1+n) \;-\; \log(1/2+b+n) \;-\; \log(1-b+n).$$

The function g_n is strictly convex on $[1/2, 1)$ with $g_n(1/2) = 0$ and

$$g_n'(b) \;=\; \frac{2b-1/2}{(1/2+b+n)(1-b+n)} \;>\; 0 \qquad \text{for all } b \in [1/2, 1).$$

Therefore, the function

$$\frac{g_n(b)}{b} \;=\; \left(\frac{b-1/2}{b}\right)\times\left(\frac{g_n(b)-g_n(1/2)}{b-1/2}\right)$$

increases on $(1/2,1)$ as the product of two positive increasing functions. Moreover, the function $a\mapsto 2ag_n(1/(2a))$ has second derivative $(1/(2a^3))g_n''(1/a)>0$ on $(1/2,1]$ and is hence convex. This implies that the mapping $a\mapsto\rho_a$ is log-convex and hence convex on $(1/2,1)$. Last, we compute

$$(a-1/2)^a\rho_a \;=\; \left(\frac{a}{\sqrt{\pi}}\,\Gamma\left(\frac{1}{2}+\frac{1}{2a}\right)\Gamma\left(2-\frac{1}{2a}\right)\right)^a \;\to\; \frac{1}{2}\quad\text{as } a\to 1/2,$$

whence the first asymptotics. Setting $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$ for the usual digamma function, we have

$$(\log\rho_a)' \;=\; \frac{1}{2a}\big(\log\rho_a \;+\; \psi(1-1/(2a)) \;-\; \psi(1/2+1/(2a))\big) \;\to\; \frac{\psi(1/2)-\psi(1)}{2} \;=\; -\log 2$$

as $a\to 1$, whence the second asymptotics. □

Remark 3.6. *The decreasing character of $a\mapsto\rho_a$ on $(1/2,\infty)$ also follows from Hölder's inequality and the following representation:*

$$\rho_a \;=\; \mathbb{E}\left[\left(\frac{\mathbf{\Gamma}_{1/2}}{\mathbf{\Gamma}_1}\right)^{\frac{1}{2a}}\right]^a$$

where $\mathbf{\Gamma}_t$ stands for the standard Gamma random variable with parameter t and the quotient inside the expectation is independent.

As a second note, applying Theorem 1.6.2 (ii) in [1] it is possible to show that

$$\log\rho_a \;=\; -\log 2 \;+\; \int_0^\infty e^{-ax}\,f(x)\,dx$$

for all $a\in(1/2,\infty)$, where

$$f(x) \quad=\quad \frac{1}{x^2}\sum_{n\geqslant 0}\left(2-e^{-\frac{x}{2n+1}}\left(1+\frac{x}{2n+1}\right)\;+\;e^{\frac{x}{2n+2}}\left(\frac{x}{2n+2}-1\right)\right)$$

is easily shown to be positive on $(0,\infty)$. This implies that the mapping $a\mapsto\rho_a$ is logarithmically completely monotone and hence completely monotone on $(1/2,\infty)$, with limit $1/e^2$ at infinity.

3.4 Density asymptotics at the logarithmic scale

We end this section with a logarithmic estimate of $\varphi(x)$ as $x\to\pm\infty$. This less precise version of Theorems 1.2 and 1.3 can be quickly derived from Theorem 3.1 and Kasahara's Tauberian theorem for densities.

Proposition 3.7. *As $x\to\infty$, one has*

$$\log\varphi(x)\;\sim\;\log\varphi(-x)\;\sim\;-(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}}.$$

Proof. Along the proof we will use the classical Mittag-Leffler function E_a defined by

$$E_a(z)\;=\;\sum_{n\geqslant 0}\frac{z^n}{\Gamma(1+an)},$$

and Mittag-Leffler random variables M_a with moment generating function $\mathbb{E}[e^{rM_a}] = E_a(r)$.

We begin the proof of Proposition 3.7 with the estimate on the positive axis. Introduce the moment generating function $\Psi(r) = \mathbb{E}[e^{rL_1}] = \sum_{n\geqslant 0}\big(\frac{\mathbb{E}[L_1^n]}{n!}\big)r^n$ as in (1.5), which is analytic on $\mathbb{C}$ by Stirling's formula combined with the equivalence

$$\frac{\mathbb{E}[L_1^n]}{n!} \sim \frac{2a\,\rho_a^n}{(a+1)\,\Gamma(1+an)}$$

given by Theorem 3.1. The latter estimate also implies

$$\Psi(r) \sim \frac{2a}{a+1}\sum_{n\geqslant 0}\frac{(\rho_a r)^n}{\Gamma(1+an)} = \frac{2a\,E_a(\rho_a r)}{a+1} \sim \frac{2\,e^{(\rho_a r)^{1/a}}}{a+1} \qquad \text{as } r\to\infty,$$

where in the second equivalence we have used the well-known estimate for the Mittag-Leffler function E_a on the positive half-line, to be found e.g. in Formula (3.4.14) of [19]. Combining this estimate on $\Psi(r)$ with Kasahara's theorem for densities – see Theorem 4.12.11 in [6] – yields after some easy algebra the required estimate

$$\log\varphi(x) \sim -(1-a)\left(\frac{a^a x}{\rho_a}\right)^{\frac{1}{1-a}} \qquad \text{as } x\to\infty.$$

We now proceed to the estimate on the negative axis. Let M_a be a Mittag-Leffler random variable and let B_a be an independent Bernoulli random variable with parameter $\frac{2a}{a+1}$. The random variable $Y_a = (\rho_a B_a)\times M_a$ has positive integer moments

$$\mathbb{E}[Y_a^n] = \frac{2a\,\rho_a^n\,n!}{(1+a)\,\Gamma(1+an)}.$$

Consider the function

$$\widetilde{\Psi}(r) = \Psi(-r) - \mathbb{E}[e^{-rY_a}] = \sum_{n\geqslant 0}\frac{\mu_n}{n!}\,r^n$$

with

$$\begin{aligned}\mu_n &= (-1)^n\left(\mathbb{E}[L_1^n] - \frac{2a\,\rho_a^n\,n!}{(1+a)\,\Gamma(1+an)}\right)\\ &= \frac{2a\kappa_a\,\rho_a^n\,(\delta_a)_n}{(a+1)\,\Gamma(1+an)}\left(1 + (-1)^n\frac{a-1}{3a+1} + \mathrm{O}\left(n^{-\delta_a}\right)\right) \asymp \frac{\rho_a^n\,(\delta_a)_n}{\Gamma(1+an)}\end{aligned} \tag{3.22}$$

by Theorem 3.1 and (3.2). Here and throughout, for two real sequences $\{u_n\}$ and $\{v_n\}$, the notation $u_n \asymp v_n$ means that there exist an index $n_0 \geqslant 1$ and two constants $c, C > 0$ such that $cv_n \leqslant u_n \leqslant Cu_n$ for all $n \geqslant n_0$. The upper and lower bounds in (3.22) follow from $0 < \frac{1-a}{3a+1} < 1$.

Introduce the following two-parameter generalization of the Mittag-Leffler function, called the Prabhakar function

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n\geqslant 0}\frac{(\gamma)_n\,z^n}{n!\,\Gamma(\beta+\alpha n)}, \tag{3.23}$$

which is analytic on $\mathbb{C}$ for all $\alpha,\beta,\gamma > 0$. Using (3.22), we deduce that

$$\log\mathbb{E}\left[e^{r|L_1|}\mathbf{1}_{\{L_1\leqslant 0\}}\right] \sim \log\widetilde{\Psi}(r) \sim \log E_{a,1}^{\delta_a}(\rho_a r) \sim (\rho_a r)^{1/a} \to \infty \quad \text{as } r\to\infty,$$

where the first equivalence is clear by the positivity of Y_a and for the third equivalence, we have used the global estimate presented in Theorem 3 of [17] – see also [29]. As in the case of the positive axis, the conclusion follows from Kasahara's Tauberian theorem for densities. $\square$

Remark 3.8. *If the sequence $\{m_n\}$ were a Hamburger moment sequence, then we would have an independent factorization*

$$L_1 \stackrel{d}{=} X \times M_a$$

where X is the real random variable having integer moments m_n, whose asymptotic behaviour would show that the right end of the support of X equals ρ_a and the left end equals $-\rho_a$. The statement of Proposition 3.7 would then follow in a more elementary way by convolution from the known asymptotics

$$f_{M_a}(x) \sim \kappa_a x^{\frac{2a-1}{2(1-a)}} e^{-(1-a)(a^a x)^{\frac{1}{1-a}}} \qquad \text{as } x \to \infty,$$

where f_{M_a} is the density function of M_a – combine e.g. Formula (14.35) and Exercise 29.18 in [24] for the latter estimate – and $\kappa_a > 0$ some constant. Unfortunately, simulations show that $\{m_n\}$ is not a Hamburger moment sequence.

4 Proof of Theorem 1.2

Strategy of the proof

In this section we prove Theorem 1.2, with the following explicit value of the prefactor c_a:

$$c_a = \sqrt{\frac{2}{\pi(1-a^2)(1+a)}} \left(\frac{a}{\rho_a}\right)^{\frac{1}{2(1-a)}}. \tag{4.1}$$

Introducing the random variable $X = -L_1/\rho_a$ with density function $\widetilde{\varphi}(x) = \rho_a \, \varphi(-\rho_a x)$ on $\mathbb{R}$, the required estimate amounts to

$$\widetilde{\varphi}(-x) \sim \sqrt{\frac{2}{\pi(1-a^2)(1+a)}} \, a^{\frac{1}{2(1-a)}} \, x^{\frac{2a-1}{2(1-a)}} \, e^{-(1-a)(a^a x)^{\frac{1}{1-a}}}, \qquad x \to \infty. \tag{4.2}$$

To prove (4.2), we will use the strong Tauberian theorem given in [15, Thm 3]. This result allows us to derive the asymptotics of the density of a random variable X, after checking three technical conditions involving the following functionals of the distribution of X:

$$\omega(r) = \mathbb{E}[e^{-rX}], \quad \xi(r) = -\frac{\omega'(r)}{\omega(r)} \quad \text{and} \quad \eta^2(r) = -\xi'(r) = \frac{\omega''(r)}{\omega(r)} - \xi^2(r). \tag{4.3}$$

We divide the proof into two steps: first, we will derive the asymptotic behaviour of the functions introduced in (4.3). In the second step, we will use these estimates to verify the three conditions of [15, Thm 3].

Detailed proof of Theorem 1.2

First step: asymptotics of the functions (4.3). Let us first note that the functions defined in (4.3) are all smooth real functions, since $\omega(r)$ is smooth and never vanishes on $\mathbb{R}$ as a moment generating function with infinite radius of convergence. As a consequence of (3.4) in Theorem 3.1 and the estimate $an\Gamma(1+an) \sim \Gamma(2+an)$, we have the expansion

$$\frac{\mathbb{E}[X^n]}{n!} = \frac{2a\,(-1)^n}{(a+1)} \left(\frac{1}{\Gamma(1+an)} + \kappa_a \frac{(\delta_a)_n((-1)^n - \frac{1-a}{3a+1})}{n!\,\Gamma(1+an)} + \mathrm{O}\left(\frac{(2\delta_a)_n}{n!\,\Gamma(2+an)} \right) \right)$$

as $n \to \infty$. Recalling the above notation (3.23) of the Prabhakar function $E^{\gamma}_{\alpha,\beta}(z)$, we deduce

$$\begin{aligned} \omega(r) &= \sum_{n\geqslant 0} \left(\frac{(-1)^n\, \mathbb{E}[X^n]}{n!} \right) r^n \\ &= \frac{2a}{a+1} \left(E_a(r) + \kappa_a \left(E^{\delta_a}_{a,1}(-r) + \frac{a-1}{3a+1}\, E^{\delta_a}_{a,1}(r) \right) \right) + \mathrm{O}\left(E^{2\delta_a}_{a,2}(r) \right) \\ &= \frac{2\, e^{r^{1/a}}}{a+1} \left(1 \,-\, \widetilde{\kappa}_a r^{-2/(a+1)} \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right), \end{aligned} \tag{4.4}$$

with the notation

$$\widetilde{\kappa}_a \;=\; \frac{\kappa_a a^{1-\delta_a}(1-a)}{3a+1}$$

and where for the equality (4.4) we have used the estimates

$$E^{\gamma}_{\alpha,\beta}(r) \;=\; \frac{r^{\frac{\gamma-\beta}{\alpha}}}{\alpha^{\gamma}}\, e^{r^{1/\alpha}} \left(1 \,+\, \mathrm{O}(r^{-1/\alpha}) \right) \qquad \text{and} \qquad E^{\gamma}_{\alpha,\beta}(-r) \;=\; \mathrm{O}(r^{-\gamma}) \qquad \text{as } r \to \infty,$$

which are valid for every $\alpha, \beta, \gamma > 0$ – see Theorem 3 in [17]. A similar argument yields

$$\frac{\mathbb{E}[X^{n+1}]}{n!} \;=\; \frac{2\,(-1)^{n+1}}{(a+1)} \left(\frac{1}{\Gamma(a+an)} \,+\, \kappa_a \frac{(\delta_a)_n \left((-1)^{n+1} - \frac{1-a}{3a+1} \right)}{n!\,\Gamma(a+an)} \,+\, \mathrm{O}\left(\frac{(2\delta_a)_n}{n!\,\Gamma(1+a+an)} \right) \right)$$

which implies, by the same estimates on the Prabhakar function,

$$\begin{aligned} \omega'(r) &= \sum_{n\geqslant 0} \left(\frac{(-1)^{n+1}\, \mathbb{E}[X^{n+1}]}{n!} \right) r^n \\ &= \frac{2a}{a+1} \left(E^1_{a,a}(r) + \kappa_a \left(E^{\delta_a}_{a,a}(-r) + \frac{a-1}{3a+1}\, E^{\delta_a}_{a,a}(r) \right) \right) + \mathrm{O}\left(E^{2\delta_a}_{a,1+a}(r) \right) \\ &= \frac{2\, r^{1/a-1}\, e^{r^{1/a}}}{a(a+1)} \left(1 \,-\, \widetilde{\kappa}_a r^{-2/(a+1)} \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right). \end{aligned} \tag{4.5}$$

Combining (4.4) and (4.5) gives the first estimate

$$\xi(r) \;=\; -\frac{r^{1/a-1}}{a} \left(1 \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right). \tag{4.6}$$

Then, after some simplifications, the details of which are left to the reader, we obtain the estimate

$$\frac{\mathbb{E}[X^{n+2}]}{n!} \;=\; \frac{2\,(-1)^n}{a(a+1)} \left(\frac{1}{\Gamma(2a-1+an)} \,+\, \kappa_a \frac{(\delta_a)_n \left((-1)^n \,-\, \frac{1-a}{3a+1} \right)}{n!\,\Gamma(2a-1+an)} \,+\, \mathrm{O}\left(\frac{(2\delta_a)_n}{n!\,\Gamma(2a+an)} \right) \right),$$

which yields similarly

$$\omega''(r) \;=\; \sum_{n\geqslant 0} \left(\frac{(-1)^n\, \mathbb{E}[X^{n+2}]}{n!} \right) r^n \;=\; \frac{2\, r^{2/a-2}\, e^{r^{1/a}}}{a^2(a+1)} \left(1 \,-\, \widetilde{\kappa}_a r^{-2/(a+1)} \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right).$$

Putting this estimate together with (4.4) and (4.6) implies

$$\eta^2(r) \;=\; \frac{r^{2/a-2}}{a^2} \left(\frac{3a\, r^{-1/a}}{2} \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right),$$

whence our final estimate

$$\eta(r) \;=\; \frac{\sqrt{1-a}\, r^{1/(2a)-1}}{a} \left(1 \,+\, \mathrm{O}\left(r^{-4/(a+1)} \right) \right),$$

concluding the first part of the proof. □

Second step: checking the assumptions of [15]. We are now in position to apply the strong Tauberian theorem of [15]. Specifically, with the notation of [15], if F stands for the distribution function of the above random variable $X = -L_1/\rho_a$, one has $R = -\infty$ and $S = \infty$ so that $\omega(r)$ satisfies (2) therein. The above asymptotics of $\eta(r)$ clearly implies the slow variation condition B in [15], and some algebra combined with the two above equivalents for $\omega(r)$ and $\eta(r)$ shows that our required (4.2) amounts to (14) therein. Hence, by Theorem 3 in [15], our proof will be complete if the domination condition (11) therein is fulfilled, and we need to establish the existence of some $g \in \mathcal{L}_1(\mathbb{R})$ such that

$$\left(\frac{r\eta(r)}{\sqrt{r^2\eta^2(r)+s^2}}\right)\left|\frac{\omega(r+\mathrm{i}s/\eta(r))}{\omega(r)}\right| \leqslant g(s)$$

for all $s \in \mathbb{R}$ and r large enough.

The estimate (4.4) remains valid in the half-plane $\{\Re(z) \geqslant 0\}$, and applying Theorem 3 in [17] to the functions $E_a(r+\mathrm{i}s/\eta(r))$, $E_{a,1}^{\delta_a}(r+\mathrm{i}s/\eta(r))$, $E_{a,1}^{\delta_a}(-r-\mathrm{i}s/\eta(r))$ and $E_{a,1}^{2\delta_a}(r+\mathrm{i}s/\eta(r))$, we obtain the following upper bound on $|\omega(r+\mathrm{i}s/\eta(r))|$:

$$\left(\frac{2\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}}\right|}{a+1} + \frac{1}{\Gamma(1-a\delta_a)}\left(\frac{\eta(r)}{\sqrt{r^2\eta^2(r)+s^2}}\right)^{\delta_a}\right)\left(1 + \mathrm{O}(r^{-2a/(a+1)})\right),$$

where the Landau symbol O does not depend on s since $|r+\mathrm{i}s/\eta(r)| \geqslant r$ for all $r \geqslant 0$ and $s \in \mathbb{R}$. We hence need to show that there exists $g \in \mathcal{L}_1(\mathbb{R})$ such that

$$\left(\frac{r\eta(r)}{\sqrt{r^2\eta^2(r)+s^2}}\right)\left(\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}-r^{1/a}}\right| + e^{-r^{1/a}}\left(\frac{\eta(r)}{\sqrt{r^2\eta^2(r)+s^2}}\right)^{\delta_a}\right) \leqslant g(s)$$

uniformly in $s \in \mathbb{R}$ for all r large enough. Since $r\eta(r) \to \infty$ and $re^{-r^{1/a}}\eta(r)^{1+\delta_a} \to 0$ as $r \to \infty$, one has

$$re^{-r^{1/a}}\left(\frac{\eta(r)}{\sqrt{r^2\eta^2(r)+s^2}}\right)^{1+\delta_a} \leqslant \left(1+s^2\right)^{-(1+\delta_a)/2} \in \mathcal{L}_1(\mathbb{R})$$

uniformly in $s \in \mathbb{R}$ for all r large enough, and we are reduced to show that there exists $r_0 > 0$ and $h \in \mathcal{L}_1(\mathbb{R})$ such that

$$\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}-r^{1/a}}\right| \leqslant h(s) \tag{4.7}$$

for all $s \in \mathbb{R}$ and $r \geqslant r_0$. Setting $z = z(s,r) = s^2/r^2\eta^2(r) \geqslant 0$ for concision, the binomial theorem implies

$$\begin{aligned}\log\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}-r^{1/a}}\right| &= r^{1/a}\sum_{n\geqslant 1}\frac{(-1/a)_{2n}}{(2n)!}(-z)^n\\ &= -\left(\frac{(1-a)z\,r^{1/a}}{a^2}\right)\sum_{n\geqslant 0}\frac{(2-1/a)_{2n}}{(2n+2)!}(-z)^n\\ &= -\left(\frac{(1-a)z\,r^{1/a}}{2a^2}\right)\sum_{n\geqslant 0}\frac{(1-1/(2a))_n(3/2-1/(2a))_n}{(3/2)_n(n+1)!}(-z)^n\end{aligned}$$

with the usual notation for the Pochhammer symbol $(x)_n$ and where in the last equality we have used the Gauss multiplication formula given e.g. at the beginning of Chapter 1.5 in [1]. Introducing the hypergeometric function

$$F_a(u) = {}_2F_1\left[\begin{matrix}1-1/(2a) & 3/2-1/(2a)\\ & 3/2\end{matrix};u\right] = \sum_{n\geqslant 0}\frac{(1-1/(2a))_n(3/2-1/(2a))_n}{(3/2)_n n!}u^n,$$

this function is analytic in $\mathbb{C}$ cut along the real axis from 1 to ∞ since $3/2 > 3/2-1/(2a) > 0$, see e.g. Theorem 2.2.1 in [1], and we have

$$\sum_{n\geqslant 0} \frac{(1-1/(2a))_n(3/2-1/(2a))_n}{(3/2)_n(n+1)!}\,(-z)^n \;=\; \frac{1}{z}\int_0^z F_a(-u)\,du.$$

Moreover, Pfaff's formula – see e.g. Theorem 2.2.5 in [1] – and the fact that $1-1/(2a) > 0$ imply

$$F_a(-u) \;=\; (1+u)^{1/(2a)-1}\,{}_2F_1\left[\begin{matrix}1-1/(2a) & 1/(2a)\\ 3/2 & \end{matrix};\frac{u}{u+1}\right] \;>\; 0$$

and

$$\begin{aligned} -F_a'(-u) \;&=\; -\left(\frac{(2a-1)(3a-1)}{3a^2}\right){}_2F_1\left[\begin{matrix}2-1/(2a) & 5/2-1/(2a)\\ 5/2 & \end{matrix};-u\right] \\ &=\; -\left(\frac{(2a-1)(3a-1)(1+u)^{1/(2a)-2}}{3a^2}\right){}_2F_1\left[\begin{matrix}2-1/(2a) & 1/(2a)\\ 5/2 & \end{matrix};\frac{u}{u+1}\right] \;<\; 0 \end{aligned}$$

for all $u\in\mathbb{R}$. Since $1/r^2\eta^2(r) = \mathrm{O}(r^{-1/a})$, there exists $r_1 > 0$ such that $z = z(s,r) < s^2$ for all $s\in\mathbb{R}$ and $r\geqslant r_1$, and by concavity we then obtain

$$\frac{1}{z}\int_0^z F_a(-u)\,du \;\geqslant\; \frac{1}{s^2}\int_0^{s^2} F_a(-u)\,du.$$

Putting everything together, we see that there exists $r_0 \geqslant r_1$ such that

$$\log\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}-\,r^{1/a}}\right| \;\leqslant\; -\left(\frac{(1-a)\,r^{1/a-2}}{2a^2\,\eta^2(r)}\right)\int_0^{s^2} F_a(-u)\,du \;\leqslant\; -\frac{1}{4}\int_0^{s^2} F_a(-u)\,du$$

for every $s\in\mathbb{R}$ and $r\geqslant r_0$, where the second inequality follows from the above equivalent for $\eta(r)$. The aforementioned Pfaff formula combined with the Gauss summation formula given e.g. in Theorem 2.2.2 of [1] show that

$$F_a(-u) \;\sim\; \left(\frac{\pi}{2\Gamma(1+1/(2a))\Gamma(3/2-1/(2a))}\right) u^{1/(2a)-1}$$

as $u\to\infty$, and integrating this equivalent finally shows that there exists $c>0$ such that

$$\log\left|e^{(r+\mathrm{i}s/\eta(r))^{1/a}-\,r^{1/a}}\right| \;\leqslant\; -c|s|^{1/a}$$

for all $s\in\mathbb{R}$ and $r\geqslant r_0$, which gives (4.7) and completes the proof. □

Remark 4.1. *In the case $a\in(1/2,a_0]$, a shorter proof of Theorem 1.2 can be provided using Corollary 2.4 and Theorem 2 in [15].*

5 Proof of Theorem 1.3

In this section we prove Theorem 1.3, with the following explicit value of the prefactor $\widehat{c}_a$:

$$\widehat{c}_a \;=\; \frac{1}{\sqrt{2\pi(1-a)}}\left(\frac{\rho_a}{a}\right)^{\frac{3a-1}{2(1-a^2)}}\frac{\rho_a^{\frac{2}{a+1}}}{2\,(a+1)^{\frac{1-a}{1+a}}\,\Gamma\left(\frac{1-a}{1+a}\right)}. \tag{5.1}$$

The proof is analogous to Theorem 1.2 except that here we need an estimate of $\xi(-r)$ and $\eta(-r)$ as $r\to\infty$, with the previous notation (4.3). Using (3.4) up to the fifth order and reasoning as in the above proof, we first obtain

$$\begin{aligned}\omega(-r) &= \frac{2a}{a+1}\left(\kappa_a\, E_{a,1}^{\delta_a}(r) \,+\, a\kappa_1^- E_{a,2}^{2\delta_a}(r) \,+\, a\kappa_2^- E_{a,2}^{\delta_a}(r) \,+\, \mathrm{O}\left(E_{a,3}^{1}(r)\right)\right)\\ &= \frac{2\, a^{\frac{2a}{a+1}}\kappa_a\, r^{-2/(a+1)}\, e^{r^{1/a}}}{a+1}\left(1 \,+\, \bar\kappa_1 r^{-2/(a+1)} \,+\, \bar\kappa_2 r^{-1/a} \,+\, \mathrm{O}\left(r^{-2/a(a+1)}\right)\right)\end{aligned}$$

with

$$\bar\kappa_1 \,=\, \frac{a^{\frac{2a}{a+1}}\kappa_1^-}{\kappa_a} \qquad \text{and} \qquad \bar\kappa_2 \,=\, \frac{a\kappa_2^-}{\kappa_a}\cdot$$

Similarly, using the expansion

$$\begin{aligned}\frac{\mathbb{E}[X^{n+1}]}{n!} \,=\, \frac{2\,(-1)^{n+1}}{(a+1)}\Bigg(&\frac{1}{\Gamma(a+an)} \,+\, \kappa_a\frac{(\delta_a)_n\left((-1)^{n+1}-\frac{1-a}{3a+1}\right)}{n!\,\Gamma(a+an)}\\ &+\frac{a\,(2\delta_a)_n\left((-1)^{n+1}\kappa_1^- + \kappa_1^+\right)}{n!\,\Gamma(1+a+an)} \,+\, \frac{a\,(\delta_a)_n\left((-1)^{n+1}\kappa_2^- + \kappa_2^+\right)}{n!\,\Gamma(1+a+an)}\\ &-\frac{2a^2\,\kappa_a\,(\delta_a)_n\left((-1)^{n+1}-\frac{1-a}{3a+1}\right)}{(a+1)\,n!\,\Gamma(1+a+an)} \,+\, \mathrm{O}\left(\frac{1}{\Gamma(2+a+an)}\right)\Bigg),\end{aligned}$$

we obtain that $\omega'(-r)$ equals

$$\frac{2}{a+1}\left(\kappa_a\, E_{a,a}^{\delta_a}(r) \,+\, a\kappa_1^- E_{a,1+a}^{2\delta_a}(r) \,+\, a\left(\kappa_2^- - \frac{2a\,\kappa_a}{a+1}\right) E_{a,1+a}^{\delta_a}(r) \,+\, \mathrm{O}\left(E_{a,2+a}^{1}(r)\right)\right).$$

The latter expression rewrites

$$\frac{2\,\kappa_a\, r^{1/a-1-2/(a+1)}\, e^{r^{1/a}}}{a^{\delta_a}(a+1)}\left(1 \,+\, \bar\kappa_1 r^{-2/(a+1)} \,+\, \bar\kappa_3 r^{-1/a} \,+\, \mathrm{O}\left(r^{-2/a(a+1)}\right)\right)$$

with

$$\bar\kappa_3 \,=\, \bar\kappa_2 \,-\, \frac{2a^2}{a+1}\cdot$$

This leads to

$$\xi(-r) \,=\, -\frac{r^{1/a-1}}{a}\left(1 \,-\, \frac{2a^2\, r^{-1/a}}{a+1} \,+\, \mathrm{O}\left(r^{-2/a(a+1)}\right)\right). \tag{5.2}$$

We next compute

$$\begin{aligned}\frac{\mathbb{E}[X^{n+2}]}{n!} \,=\,=\, \frac{2\,(-1)^{n}}{a(a+1)}\Bigg(&\frac{1}{\Gamma(2a-1+an)} \,+\, \kappa_a\,\frac{(\delta_a)_n\left((-1)^{n} \,-\, \frac{1-a}{3a+1}\right)}{n!\,\Gamma(2a-1+an)}\\ &+\frac{a\,(2\delta_a)_n\left((-1)^{n}\kappa_1^- + \kappa_1^+\right)}{n!\,\Gamma(2a+an)} \,+\, \frac{a\,(\delta_a)_n\left((-1)^{n}\kappa_2^- + \kappa_2^+\right)}{n!\,\Gamma(2a+an)}\\ &+\frac{(1-5a^2)\,\kappa_a\,(\delta_a)_n\left((-1)^{n}-\frac{1-a}{3a+1}\right)}{(a+1)\,n!\,\Gamma(2a+an)} \,+\, \mathrm{O}\left(\frac{1}{\Gamma(1+2a+an)}\right)\Bigg),\end{aligned}$$

which implies

$$\begin{aligned}\omega''(-r) &= \frac{2}{a(a+1)}\Bigg(\kappa_a\, E_{a,2a-1}^{\delta_a}(r) \,+\, a\kappa_1^- E_{a,2a}^{2\delta_a}(r)\\ &\qquad +\left(a\kappa_2^- + \frac{(1-5a^2)\,\kappa_a}{a+1}\right) E_{a,2a}^{\delta_a}(r) \,+\, \mathrm{O}\left(E_{a,1+2a}^{1}(r)\right)\Bigg)\\ &= \frac{2\,\kappa_a\, r^{2/a-2-2/(a+1)}\, e^{r^{1/a}}}{a^{1+\delta_a}(a+1)}\left(1 \,+\, \bar\kappa_1 r^{-2/(a+1)} \,+\, \bar\kappa_4 r^{-1/a} \,+\, \mathrm{O}\left(r^{-2/a(a+1)}\right)\right)\end{aligned}$$

with

$$\bar{\kappa}_4 \,=\, \bar{\kappa}_2 \,+\, \frac{1-5a^2}{a+1}\cdot$$

This leads to

$$\frac{\omega''(-r)}{\omega(-r)} \,=\, \frac{r^{2/a-2}}{a^2}\left(1 \,+\, \frac{(1-5a^2)\,r^{-1/a}}{a+1} \,+\, \mathrm{O}\left(r^{-2/a(a+1)}\right)\right)$$

and finally, combining this estimate with (5.2), to

$$\eta(-r) \,=\, \sqrt{\frac{\omega''(-r)}{\omega(-r)} \,-\, \xi^2(-r)} \,=\, \frac{\sqrt{1-a}\,r^{1/(2a)-1}}{a}\left(1 \,+\, \mathrm{O}\left(r^{-(1-a)/a(a+1)}\right)\right). \tag{5.3}$$

Similarly as on the positive axis, we can then apply Theorem 3 in [15] and obtain, with the previous notation for the function $\widetilde{\varphi}$, the estimate

$$\widetilde{\varphi}(x) \,\sim\, \left(\frac{a^{\frac{1-3a}{2(1-a^2)}}\,\rho_a^{\frac{2}{a+1}}}{\sqrt{8\pi(1-a)}(a+1)^{\delta_a}\,\Gamma\left(\frac{1-a}{1+a}\right)}\right) x^{\frac{2a^2-3a-1}{2(1-a^2)}}\, e^{-(1-a)(a^a x)^{\frac{1}{1-a}}}, \qquad x\to\infty,$$

which completes the proof. □

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