



On the distribution of cumulative Parisian ruin



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ABSTRACT

We introduce the concept of cumulative Parisian ruin, which is based on the time spent in the red by the underlying surplus process. Our main result is an explicit representation for the distribution of the occupation time, over a finite-time horizon, for a compound Poisson process with drift and exponential claims. The Brownian ruin model is also studied in details. Finally, we analyse for a general framework the relationships between cumulative Parisian ruin and classical ruin, as well as with Parisian ruin based on exponential implementation delays.

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1. Introduction

Historically, in classical actuarial ruin theory, the central topic has been the analysis of the probability of ruin, which is a measure to assess the overall risk of an insurance company. Ruin occurs when the surplus process falls below a certain threshold level for the first time, e.g. the insurer's minimum capital requirement set by the regulatory body. Then, more sophisticated ruin-based risk measures using the timing and the severity of such a capital shortfall have been proposed, culminating with so-called Gerber–Shiu distributions; see Gerber and Shiu (1998) and Kyprianou (2013).

However, as discussed in dos Reis (1993): “[...] sometimes the event ruin has a very small probability and the portfolio is just one out of many existing in the company. The company can have enough funds available to support (or ask for external support) some negative surplus for some time, in the hope that the portfolio will recover in the future, allowing the company to keep this business alive. This can be regarded as an investment, since the process will recover in the future.” This is why, in recent years, new ruin models, incorporating this realistic feature that a company is not immediately liquidated when it defaults, have been proposed: models based on Parisian ruin (see, e.g., Czarna and Palmowski, 2011; Loeffen et al., 2013; Landriault et al., 2014; Baurdoux et al., 2016) or random observations (see, e.g., Albrecher et al., 2011,

2013) and Omega models (see, e.g., Albrecher et al., 2011; Gerber et al., 2012 and Albrecher and Lautscham, 2013). In those papers, new definitions of ruin, bankruptcy and/or liquidation are proposed and studied.

In this direction, we will introduce the concept of cumulative Parisian ruin, which is based on the time spent in the red by the underlying surplus process. While Parisian ruin has been inspired by Parisian options, the idea of cumulative Parisian ruin comes from cumulative Parisian options, another type of occupation-time options. In the definition of Parisian ruin (the duration of) periods of financial distress are considered separately, while cumulative Parisian ruin considers the aggregation of those durations. As expected, this notion of ruin is closely related to the occupation-time process of the underlying surplus process in the model. We will also analyse the relationships between cumulative Parisian ruin and classical ruin, as well as with Parisian ruin based on exponential implementation delays.

1.1. Insurance risk models

In this paper, we will mainly consider the following two classical actuarial ruin models: the Cramér–Lundberg model with exponential claims and the Brownian ruin model. First, in the Cramér–Lundberg model, the surplus of an insurance company/portfolio is modelled by a compound Poisson process with drift $X^{\text{CL}} = \{X_t^{\text{CL}}, t \geq 0\}$, that is

$$X_t^{\text{CL}} = X_0^{\text{CL}} + ct - \sum_{i=1}^{N_t} C_i, \quad (1)$$

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where $c > 0$ denotes the constant premium rate, $N = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and the C_i 's form a sequence of independent and identically distributed random variables (also independent of N) representing the claim amounts; each C_i is assumed to follow an exponential distribution with rate $\alpha > 0$. In this model, the random variable $\sum_{i=1}^{N_t} C_i$ represents the aggregate claim payments made up to time t .

On the other hand, in the Brownian ruin model, the surplus of the insurance company is modelled by a Brownian motion with drift $X_t^{\text{BM}} = \{X_t^{\text{BM}}, t \geq 0\}$, that is

$$X_t^{\text{BM}} = X_0^{\text{BM}} + ct + \sigma B_t, \quad (2)$$

where $c > 0$ denotes the constant premium rate, $\sigma > 0$ and $B = \{B_t, t \geq 0\}$ is a standard Brownian motion.

Note that both surplus processes X^{CL} and X^{BM} are special cases of a spectrally negative Lévy process (SNLP), also known as a general Lévy insurance risk process, that is a càdlàg process $X = \{X_t, t \geq 0\}$ with stationary and independent increments and no positive jumps. There is a rich literature on this family of stochastic processes for tackling classical ruin problems; see e.g. Kyprianou (2013, 2014).

1.2. Definitions of ruin and bankruptcy

Let $X = \{X_t, t \geq 0\}$ be a surplus process. In what follows, the law of X with initial capital $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

The time of classical ruin is defined as

$$\tau_0^- = \inf \{t > 0: X_t < 0\},$$

yielding the classical ruin-based risk measure $\mathbb{P}_x(\tau_0^- < \infty)$, i.e. the probability of ruin. From a practical point of view, it is more interesting to consider the finite-time probability of ruin $\mathbb{P}_x(\tau_0^- \leq t)$, where $t > 0$ is a fixed deterministic time corresponding to the time horizon of interest for the risk managers.

In the last few years, in order to distinguish between default and bankruptcy, several concepts of ruin have been proposed. In particular, the idea of Parisian ruin has attracted a lot of attention; it considers the application of an implementation delay in the recognition of an insurer's capital insufficiency. More precisely, it assumes that Parisian ruin occurs if the excursion below the critical threshold level 0 is longer than a pre-determined time called the implementation delay, or the *clock*. Two types of Parisian ruin have been considered: with deterministic delays or with stochastic delays.

First, in Czarna and Palmowski (2011) and Loeffen et al. (2013), a Parisian ruin time (with a deterministic delay $r > 0$) is defined as

$$\tau_r = \inf \{t > 0: t - g_t > r\},$$

where $g_t = \sup \{0 \leq s \leq t: X_s \geq 0\}$, and studied for Lévy insurance risk models. The stopping time τ_r is the first time the surplus process has stayed below zero for a consecutive period of length greater than the fixed implementation delay r . If it is finite, then at time τ_r the company is said to be *Parisian ruined*. We will call this *deterministic Parisian ruin*, where *deterministic* refers to the implementation delay.

On the other hand, in Landriault et al. (2011, 2014) and Baurdoux et al. (2016), exponentially distributed delays have been considered. A descriptive definition of Parisian ruin with exponential delays is given in Baurdoux et al. (2016), for spectrally negative Lévy processes. The idea is to mark the Poisson point process of excursions away from 0 by independent copies of a generic exponential random variable $\mathbf{e}_q$ with rate $q > 0$. Then, the Parisian ruin time is defined as

$$\kappa_q = \inf \{t > 0: t - g_t > \mathbf{e}_q^{g_t}\},$$

where $g_t = \sup \{0 \leq s \leq t: X_s \geq 0\}$ and where each random variable $\mathbf{e}_q^{g_t}$ is exponentially distributed with rate $q > 0$. We will call this *exponential Parisian ruin*.

Note that for the two definitions of Parisian ruin mentioned above, there is a different clock for each excursion, either deterministic or stochastic.

Remark 1.1. Note that the definition of exponential Parisian ruin corresponds to a specific choice of function $\omega(\cdot)$ in an Omega model; see Gerber et al. (2012), Albrecher and Loutscham (2013) and Loeffen et al. (2014).

The rest of the paper is organized as follows. In Section 2, we introduce the concept of cumulative Parisian ruin. In Section 3, we derive our main results. More precisely, in Section 3.1 and in Section 3.2, we obtain the distribution of cumulative Parisian ruin when the underlying surplus process is a Brownian motion risk process and a Cramér–Lundberg process with exponential claims, respectively. Section 3.3 is devoted to the study of the relationships between cumulative Parisian and classical ruin, as well as with exponential Parisian ruin.

2. Cumulative Parisian ruin

We now propose a new definition of actuarial ruin based on the occupation-time process of the surplus process X , namely the process

$$t \mapsto \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds.$$

We define the time of cumulative Parisian ruin (at level $r > 0$) by

$$\sigma_r = \inf \left\{ t > 0: \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds > r \right\}. \quad (3)$$

Remark 2.1. Note that for cumulative Parisian ruin, the parameter r could be interpreted as the length of a clock started at the beginning of the first excursion, paused when the process returns above zero, and resumed at the beginning of the next excursion, and so on. In other words, it is the length of this single clock that is compared to the sum of all excursions below zero, as opposed to the previous definitions of Parisian ruin where a new clock is started at the beginning of each excursion.

Remark 2.2. This definition of ruin is referred to as cumulative Parisian ruin due to its ties with cumulative Parisian options; see e.g. Hugonnier (1999). Note also that this definition of bankruptcy has been used as a definition of default in structural credit risk models; see Yildirim (2006) and Makarov et al. (2015).

We are interested in the finite-time probability of cumulative Parisian ruin

$$\mathbb{P}_x(\sigma_r \leq t),$$

for each fixed deterministic time $t > 0$; we are also interested in the infinite-time horizon version, that is

$$\mathbb{P}_x(\sigma_r < \infty).$$

One motivation for deriving the distribution of the cumulative Parisian ruin time is the possibility of defining a new VaR-type risk measure as in the works of J. Trufin and his co-authors (Trufin et al., 2011; Loisel and Trufin, 2014; Mitric and Trufin, 2016). More precisely, for a given bound $\varepsilon > 0$, set

$$\rho_\varepsilon(X, t) := \inf \{x \geq 0: \mathbb{P}_x(\sigma_r \leq t) \leq \varepsilon\}$$

where the time horizon $0 < t \leq \infty$ and the model parameters are fixed. This ruin-based risk measure is the amount of initial

capital needed in order to bound the probability of cumulative Parisian ruin by ε . For more on the underlying idea for the above definition, the reader is referred to [Trufin et al. \(2011\)](#), where a detailed motivation is provided.

The goal of this paper is to provide a new methodological tool, namely the probability of cumulative Parisian ruin, for risk management purposes. For example, our results make possible the study of a risk measure such as $\rho_\varepsilon(X, t)$, its applicability in risk management as well as a comparison with other ruin-related risk measures that can be found in the literature; this is left for future research. Lastly, it should be emphasized here that our main results allow for both finite-time and infinite-time analyses.

3. Distribution of the occupation time

Clearly, for fixed values $t, r > 0$, we have

$$\{\sigma_r > t\} = \left\{ \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq r \right\},$$

so, the finite-time probability of cumulative Parisian ruin is given by

$$\begin{aligned} \mathbb{P}_x(\sigma_r \leq t) &= 1 - \mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq r\right) \\ &= 1 - \int_0^r \mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds \in du\right). \end{aligned} \quad (4)$$

Then, we also have

$$\begin{aligned} \mathbb{P}_x(\sigma_r < \infty) &= 1 - \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq r\right) \\ &= 1 - \int_0^r \mathbb{P}_x\left(\int_0^\infty \mathbb{1}_{(-\infty, 0)}(X_s) ds \in du\right). \end{aligned}$$

As the distribution of cumulative Parisian ruin is closely related to the distribution of the occupation-time process associated with the underlying surplus process X over a finite time horizon, we will now study the latter for the classical surplus processes. While there is a vast literature on the analysis of occupation times related to Brownian motion, it turns out that little is known about the distribution of the occupation-time process for a compound Poisson process with drift over a finite time horizon.

Of course, it is very difficult to obtain analytical expressions for the probability of finite-time cumulative Parisian ruin for a general insurance risk process. However, there is a quite simple expression of its Laplace transform when the initial capital is zero in the case of spectrally negative Lévy processes.

Proposition 3.1. *If X is a SNLP such that $\mathbb{E}[X_1] = \psi'(0+) > 0$, then, for $p, q > 0$,*

$$\int_0^\infty e^{-pt} \mathbb{E}\left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds}\right] dt = \frac{\Phi(p+q)}{p+q} \frac{1}{\Phi(p)}, \quad (5)$$

where $\Phi(q)$ is the largest root of Lundberg's equation $\psi(\theta) - q = 0$, for a fixed $q > 0$, and where $\psi(\theta) := \ln(\mathbb{E}[e^{\theta X_1}])$.

Proof. See e.g. Remark 4.1 in [Landriault et al. \(2011\)](#). $\square$

Since

$$\begin{aligned} &\int_0^\infty e^{-pt} \mathbb{E}\left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds}\right] dt \\ &= \int_0^\infty \int_0^\infty e^{-pt-qu} \mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds \in du\right) dt, \end{aligned}$$

we obtain the distribution of the occupation time by inverting this double Laplace transform, which is our objective for the two main ruin models considered in this paper.

3.1. Brownian ruin model

One of Paul Lévy's arcsine laws gives the distribution of the occupation time of the positive/negative half-line for a standard Brownian motion $B = \{B_t, t \geq 0\}$. More precisely, for $0 < s < t$,

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(B_u) du \leq s\right) = \frac{2}{\pi} \arcsin\left(\sqrt{\frac{s}{t}}\right).$$

This result was then extended to a Brownian motion with drift by [Akahori \(1995\)](#) and [Takács \(1996\)](#).

Recall that in the Brownian ruin model of Eq. (2), the surplus is given by

$$X_t^{\text{BM}} = X_0^{\text{BM}} + ct + \sigma B_t,$$

where $c > 0$ and $\sigma > 0$. In this case, $\psi(\theta) = c\theta + \frac{\sigma^2}{2}\theta^2$, with right-inverse $\Phi(q) = \frac{1}{\sigma^2}(\sqrt{c^2 + 2\sigma^2 q} - c)$.

From [Proposition 3.1](#), the double Laplace transform of the cumulative Parisian ruin σ_r is given by, for $p, q > 0$,

$$\begin{aligned} &\int_0^\infty e^{-pt} \mathbb{E}\left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s^{\text{BM}}) ds}\right] dt \\ &= \frac{\sqrt{c^2 + 2\sigma^2(p+q)} - c}{p+q} \frac{1}{\sqrt{c^2 + 2\sigma^2 p} - c} \\ &= \frac{1}{2\sigma^2 p(p+q)} \left(\sqrt{c^2 + 2\sigma^2(p+q)} - c\right) \left(\sqrt{c^2 + 2\sigma^2 p} + c\right). \end{aligned}$$

We recognize (1.4) in [Akahori \(1995\)](#) (given for the case $\sigma = 1$) and then, by Laplace inversion, we recover the generalized arcsine law of the occupation time of the negative half-line for a Brownian motion with drift.

Theorem 1 ([Akahori, 1995](#)). *For a fixed $t > 0$, the distribution of the occupation time of the negative half-line, when $X_0^{\text{BM}} = 0$, is given by*

$$\begin{aligned} &\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u^{\text{BM}}) du \in ds\right) \\ &= \frac{2}{\sigma^2} \left\{ \frac{\sigma e^{-(c^2/2\sigma^2)s}}{\sqrt{2\pi s}} - c\bar{N}\left(\frac{c\sqrt{s}}{\sigma}\right) \right\} \\ &\quad \times \left\{ c + \frac{\sigma e^{-(c^2/2\sigma^2)(t-s)}}{\sqrt{2\pi(t-s)}} - c\bar{N}\left(\frac{c\sqrt{t-s}}{\sigma}\right) \right\} ds, \end{aligned}$$

for $0 \leq s \leq t$, where $\bar{N}$ denotes the tail of the standard normal distribution, that is

$$\bar{N}(x) = \int_x^\infty \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy.$$

Clearly, using Eq. (4), we have an explicit expression for $\mathbb{P}(\sigma_r \leq t)$ in the Brownian model. Using the continuity of Brownian motion, we can easily obtain an explicit expression for $\mathbb{P}_x(\sigma_r \leq t)$, i.e. when $X_0^{\text{BM}} = x > 0$. Indeed, using the strong Markov property and the continuity of X^{BM} , we easily deduce that, for $s \leq t$,

$$\begin{aligned} &\mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u^{\text{BM}}) du \leq s\right) \\ &= \mathbb{P}_x(\tau_0^- > t-s) + \int_0^{t-s} \mathbb{P}_x(\tau_0^- \in du) \\ &\quad \times \mathbb{P}\left(\int_0^{t-u} \mathbb{1}_{(-\infty, 0)}(X_v^{\text{BM}}) dv \leq s\right), \end{aligned}$$

where it is well known (see e.g. Karatzas and Shreve, 1991) that

$$\mathbb{P}_x(\tau_0^- \in du) = \frac{x}{\sqrt{2\pi\sigma^2u^3}} \exp\left\{-\frac{(x+cu)^2}{2\sigma^2u}\right\} du.$$

From the above, we easily deduce the following expression for the probability of cumulative Parisian ruin:

$$\begin{aligned} \mathbb{P}_x(\sigma_r \leq t) &= 1 - \mathbb{P}_x\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u^{\text{BM}}) du \leq r\right) \\ &= \int_0^{t-r} \mathbb{P}\left(\int_0^{t-u} \mathbb{1}_{(-\infty,0)}(X_s^{\text{BM}}) ds > r\right) \\ &\quad \times \frac{x}{\sqrt{2\pi\sigma^2u^3}} \exp\left\{-\frac{(x+cu)^2}{2\sigma^2u}\right\} du. \end{aligned}$$

3.2. Cramér–Lundberg model with exponential claims

Recall that in the Cramér–Lundberg model with exponential claims introduced in Eq. (1), the surplus is given by

$$X_t^{\text{CL}} = X_0^{\text{CL}} + ct - \sum_{i=1}^{N_t} C_i,$$

where $c > 0$, $N = \{N_t, t \geq 0\}$ is a Poisson process with rate $\lambda > 0$ and the C_i 's form a sequence of independent and identically distributed random variables following an exponential distribution with rate $\alpha > 0$.

In this model, for $p \geq 0$, Lundberg's equation takes the form $\psi(\theta) - p = 0$, where $\psi(\theta) = c\theta - \frac{\lambda\theta}{\alpha+\theta}$, and its solutions are given by

$$\Phi(p) = \frac{1}{2c} (p + \lambda - c\alpha + \sqrt{\Delta_p}),$$

$$\theta(p) = \frac{1}{2c} (p + \lambda - c\alpha - \sqrt{\Delta_p}),$$

where $\Delta_p = (p + \lambda - c\alpha)^2 + 4c\alpha p$. Note that we can also write $\Delta_p = (p + \lambda + c\alpha)^2 - 4c\alpha\lambda$.

Clearly, for a fixed $t > 0$, the distribution

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u^{\text{CL}}) du \in ds\right)$$

has a mass a_t at 0, which is the survival probability $\mathbb{P}(\tau_0^- > t)$. We now derive from Proposition 3.1 an expression for this distribution.

Proposition 3.2. For a fixed $t > 0$, we have

$$\begin{aligned} \mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u^{\text{CL}}) du \in ds\right) \\ = a_t \delta_0(ds) + a_{t-s} (\lambda - c\alpha(1 - a_s)) \mathbb{1}_{(0,t)}(s) ds, \end{aligned} \quad (6)$$

with

$$\begin{aligned} a_t &= \left(1 - \frac{\lambda}{c\alpha}\right)_+ + \frac{2\lambda}{\pi} e^{-(\lambda+c\alpha)t} \int_{-1}^1 \frac{\sqrt{1-u^2}}{\lambda + c\alpha + 2\sqrt{c\alpha\lambda}u} \\ &\quad \times e^{-2\sqrt{c\alpha\lambda}tu} du, \end{aligned} \quad (7)$$

and $z_+ := \max(z, 0)$.

Note that there exists other equivalent expressions for $a_t = \mathbb{P}(\tau_0^- > t)$ in the Cramér–Lundberg model with exponential claims. See e.g. Drekić (2009), or Asmussen and Albrecher (2010, Proposition 1.3) and the references therein.

Proof. For simplicity, we will use X instead of X^{CL} in the proof. From Proposition 3.1, we know that the double Laplace transform of the occupation time of the negative half-line is equal to

$$\int_0^\infty e^{-pt} \mathbb{E}\left[e^{-q \int_0^t \mathbb{1}_{(-\infty,0)}(X_s) ds}\right] dt = \frac{\Phi(p+q)}{p+q} \frac{1}{\Phi(p)}. \quad (8)$$

Since $\Phi(v)/v$ converges to $1/c$ when v goes to ∞ , we deduce that (8) converges to $(c\Phi(p))^{-1}$ when q goes to ∞ . As mentioned above, this is due to the fact that $\int_0^t \mathbb{1}_{(-\infty,0)}(X_u) du$ has a mass at 0, which means that its distribution under $\mathbb{P}$ can be written in the following way:

$$\mathbb{P}\left(\int_0^t \mathbb{1}_{(-\infty,0)}(X_u) du \in ds\right) = a_t \delta_0(ds) + h_t(ds),$$

where $a_t \in [0, 1]$ and $h_t(ds)$ is a nonnegative measure on $[0, t]$ with no mass at zero.

We easily see that

$$\mathcal{L}a(p) := \int_0^\infty e^{-pt} a_t dt = \int_0^\infty e^{-pt} \mathbb{P}(\tau_0^- > t) dt = \frac{1}{c\Phi(p)}. \quad (9)$$

Then, by linearity, the double Laplace transform of $(t, s) \mapsto h_t(ds)$ must be equal to

$$\begin{aligned} \mathcal{L}h(p, q) &= \frac{1}{c\Phi(p)} \left(\frac{c\Phi(p+q)}{p+q} - 1\right) \\ &= \frac{1}{c\Phi(p)} \left(-\frac{c\theta(p+q)}{p+q} + \frac{\lambda - c\alpha}{p+q}\right) \\ &= \frac{1}{c\Phi(p)} \left(\frac{\alpha}{\Phi(p+q)} + \frac{\lambda - c\alpha}{p+q}\right), \end{aligned} \quad (10)$$

where the second equality is obtained using the explicit expressions for $\Phi(\cdot)$ and $\theta(\cdot)$, while the third equality is a consequence of the following identity: $c\Phi(p+q)\theta(p+q) = -\alpha(p+q)$.

Using basic properties of Laplace transforms and Eq. (9), the expression of the Laplace transform of $(t, s) \mapsto h_t(ds)$ implies that

$$h_t(ds) = a_{t-s}(c\alpha a_s + \lambda - c\alpha) \mathbb{1}_{(0,t)}(s) ds.$$

Consequently, to identify the law of $\int_0^t \mathbb{1}_{(-\infty,0)}(X_u) du$, we need to identify a_t . Instead of using one of the available expressions from the literature, we derive a new expression.

First, using the conjugate expression of square roots and the explicit form of $\Phi(p)$, we notice that

$$\begin{aligned} \mathcal{L}a(p) &= \frac{\sqrt{(p+\lambda+c\alpha)^2 - 4c\alpha\lambda} - (p+\lambda+c\alpha)}{2c\alpha p} + \frac{1}{p} \\ &= \frac{-2\lambda}{p\left(\sqrt{(p+\lambda+c\alpha)^2 - 4c\alpha\lambda} + (p+\lambda+c\alpha)\right)} + \frac{1}{p}. \end{aligned}$$

As, for functions f and g , $\mathcal{L}f(p) = \frac{1}{p} \int_0^p \mathcal{L}g(q) dq$ implies $f(t) = \int_t^\infty s^{-1} g(s) ds$ and the Laplace transform of $t \mapsto I_1(kt)$ is equal to $p \mapsto k\left(\sqrt{p^2 - k^2} + p\right)^{-1} (p^2 - k^2)^{-1/2}$ with I_1 the modified Bessel function of the first kind of order 1 (see e.g. Polyanin and Manzhirov, 2008, Supplement 4 for more details), we deduce that, when $\lambda \geq c\alpha$, or equivalently $\mathbb{E}[X_1] \leq 0$,

$$a_t = \sqrt{\frac{\lambda}{c\alpha}} \int_t^\infty \frac{1}{s} e^{-(\lambda+c\alpha)s} I_1(2\sqrt{c\alpha\lambda}s) ds,$$

and, when $\lambda < c\alpha$, or equivalently $\mathbb{E}[X_1] > 0$,

$$a_t = 1 - \frac{\lambda}{c\alpha} + \sqrt{\frac{\lambda}{c\alpha}} \int_t^\infty \frac{1}{s} e^{-(\lambda+c\alpha)s} I_1(2\sqrt{c\alpha\lambda}s) ds.$$

At last, the final result is obtained using Fubini's theorem and the following integral form for the modified Bessel function: for $s > 0$,

$$I_1(s) = \frac{s}{\pi} \int_{-1}^1 e^{-su} \sqrt{1-u^2} du. \quad \square$$

As opposed to the Brownian case, it is more delicate to obtain the corresponding distribution under $\mathbb{P}_x$ for $x > 0$, i.e. when $X_0^{\text{CL}} = x > 0$, since $X_{\tau_0^-} < 0$ a.s.

Again note that, for a fixed $t > 0$, the distribution

$$\mathbb{P}_x \left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u^{\text{CL}}) du \in ds \right)$$

has a mass a_t^x at 0, which is the survival probability $\mathbb{P}_x(\tau_0^- > t)$.

Theorem 2. For a fixed $t > 0$, we have

$$\mathbb{P}_x \left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u^{\text{CL}}) du \in ds \right) = a_t^x \delta_0(ds) + (a_{t-s}^x + k_{t-s}^x) (\lambda - c\alpha (1 - a_s^0)) \mathbb{1}_{(0,t)}(s) ds, \quad (11)$$

with

$$\begin{aligned} a_t^x &= 1 - \lambda e^{-\alpha x} \int_0^t e^{-(\lambda+c\alpha)s} \left[I_0 \left(2\sqrt{\lambda c \alpha} \sqrt{s(s+x/c)} \right) \right. \\ &\quad \left. - \frac{s}{s+x/c} I_2 \left(2\sqrt{\lambda c \alpha} \sqrt{s(s+x/c)} \right) \right] ds, \\ k_t^x &= e^{-\alpha x} - 1 \\ &\quad + \lambda \alpha \lambda e^{-\alpha x} \int_0^t e^{-(\lambda+c\alpha)s} \left[I_0 \left(2\sqrt{c \alpha \lambda} \sqrt{s(s+x/c)} \right) \right. \\ &\quad \left. - I_2 \left(2\sqrt{c \alpha \lambda} \sqrt{s(s+x/c)} \right) \right] ds, \end{aligned} \quad (12)$$

where I_ν represents the modified Bessel function of the first kind of order ν , which can be written as: for $s \geq 0$,

$$\begin{aligned} I_0(s) &= \frac{1}{\pi} \int_{-1}^1 e^{-su} (1-u^2)^{-1/2} du \quad \text{and} \\ I_2(s) &= \frac{s^2}{3\pi} \int_{-1}^1 e^{-su} (1-u^2)^{3/2} du. \end{aligned}$$

Proof. For simplicity, we will use X instead of X^{CL} in the proof. First, note that we have the following relationships between $\Phi(\cdot)$ and $\theta(\cdot)$: for $p, q \geq 0$,

$$\Phi(p) - \theta(p) = \frac{1}{c} \sqrt{\Delta_p}, \quad (13)$$

$$\Phi(p)\theta(p) = -\frac{\alpha p}{c}, \quad (14)$$

$$(\theta(p+q) - \Phi(p))(\theta(p+q) - \theta(p)) = \frac{q}{c} (\theta(p+q) + \alpha). \quad (15)$$

From Guérin and Renaud (2016, Corollary 1), for $x \in \mathbb{R}$ and $p, q > 0$, we deduce an explicit expression for the double Laplace transform of the occupation time of the negative half-line

$$\begin{aligned} &\int_0^\infty e^{-pt} \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds} \right] dt \\ &= \int_{-\infty}^\infty \left\{ \left(\frac{\Phi(p+q) - \Phi(p)}{q} \right) \mathcal{H}^{(p+q, -q)}(x) \mathcal{H}^{(p, q)}(-y) \right. \\ &\quad \left. - \mathcal{W}_x^{(p, q)}(x-y) \right\} dy, \end{aligned}$$

where, in the case of a Cramér–Lundberg risk process with exponential claims (see e.g. Guérin and Renaud, 2016 for more details), for $x \geq 0$,

$$\mathcal{H}^{(p, q)}(x) = \frac{q}{\sqrt{\Delta_{p+q}}} \left[\frac{\alpha + \Phi(p+q)}{\Phi(p+q) - \Phi(p)} e^{\Phi(p+q)x} - \frac{\alpha + \theta(p+q)}{\theta(p+q) - \Phi(p)} e^{\theta(p+q)x} \right]$$

and

$$\mathcal{H}^{(p+q, -q)}(x) = \frac{q}{\sqrt{\Delta_p}} \left(\frac{\alpha + \Phi(p)}{\Phi(p+q) - \Phi(p)} e^{\Phi(p)x} - \frac{\alpha + \theta(p)}{\Phi(p+q) - \theta(p)} e^{\theta(p)x} \right),$$

and where, for $x < 0$, $\mathcal{H}^{(p, q)}(x) = \exp(\Phi(p)x)$ and $\mathcal{H}^{(p+q, -q)}(x) = \exp(\Phi(p+q)x)$. Also, for $x \geq a$,

$$\begin{aligned} \mathcal{W}_a^{(p, q)}(x) &= q \frac{\alpha + \Phi(p+q)}{\sqrt{\Delta_{p+q} \Delta_p}} e^{\Phi(p+q)(x-a)} \\ &\quad \times \left[\frac{\alpha + \Phi(p)}{\Phi(p+q) - \Phi(p)} e^{\Phi(p)a} - \frac{\alpha + \theta(p)}{\Phi(p+q) - \theta(p)} e^{\theta(p)a} \right] \\ &\quad - q \frac{\alpha + \theta(p+q)}{\sqrt{\Delta_{p+q} \Delta_p}} e^{\theta(p+q)(x-a)} \\ &\quad \times \left[\frac{\alpha + \Phi(p)}{\theta(p+q) - \Phi(p)} e^{\Phi(p)a} - \frac{\alpha + \theta(p)}{\theta(p+q) - \theta(p)} e^{\theta(p)a} \right], \end{aligned}$$

for $x \in [0, a]$, $\mathcal{W}_a^{(p, q)}(x) = W^{(p)}(a)$, and finally, for $x < 0$, $\mathcal{W}_a^{(p, q)}(x) = 0$. Therefore, for $y < 0$, we have

$$\begin{aligned} \mathcal{W}_x^{(p, q)}(x-y) &= \frac{\alpha + \Phi(p+q)}{\sqrt{\Delta_{p+q}}} \mathcal{H}^{(p+q, -q)}(x) e^{-\Phi(p+q)y} \\ &\quad - q \frac{\alpha + \theta(p+q)}{\sqrt{\Delta_{p+q} \Delta_p}} \left[\frac{\alpha + \Phi(p)}{\theta(p+q) - \Phi(p)} e^{\Phi(p)x} \right. \\ &\quad \left. - \frac{\alpha + \theta(p)}{\theta(p+q) - \theta(p)} e^{\theta(p)x} \right] e^{-\theta(p+q)y}. \end{aligned}$$

We divide the computation of the integral into three parts. We first consider the integral on the negative half-line. For $y < 0$, using (15) and (13), we have

$$\begin{aligned} &\left(\frac{\Phi(p+q) - \Phi(p)}{q} \right) \mathcal{H}^{(p+q, -q)}(x) \mathcal{H}^{(p, q)}(-y) - \mathcal{W}_x^{(p, q)}(x-y) \\ &= \frac{\alpha + \theta(p+q)}{\sqrt{\Delta_{p+q}}} \left[-\frac{\Phi(p+q) - \Phi(p)}{\theta(p+q) - \Phi(p)} \mathcal{H}^{(p+q, -q)}(x) \right. \\ &\quad \left. + \frac{q}{\sqrt{\Delta_p}} \left(\frac{\alpha + \Phi(p)}{\theta(p+q) - \Phi(p)} e^{\Phi(p)x} - \frac{\alpha + \theta(p)}{\theta(p+q) - \theta(p)} e^{\theta(p)x} \right) \right] e^{-\theta(p+q)y} \\ &= \frac{c(\alpha + \theta(p))}{\sqrt{\Delta_p \Delta_{p+q}}} \left[(\Phi(p+q) - \Phi(p)) \frac{(\theta(p+q) - \theta(p))}{\Phi(p+q) - \theta(p)} \right. \\ &\quad \left. - (\theta(p+q) - \Phi(p)) \right] e^{\theta(p)x} e^{-\theta(p+q)y} \\ &= \frac{\alpha + \theta(p)}{c(\Phi(p+q) - \theta(p))} e^{\theta(p)x} e^{-\theta(p+q)y}. \end{aligned}$$

Consequently,

$$\begin{aligned} & \int_{-\infty}^0 \left\{ \left(\frac{\Phi(p+q) - \Phi(p)}{q} \right) \mathcal{H}^{(p+q, -q)}(x) \mathcal{H}^{(p, q)}(-y) \right. \\ & \quad \left. - \mathcal{W}_x^{(p, q)}(x-y) \right\} dy \\ &= -\frac{\alpha + \theta(p)}{c\theta(p+q)(\Phi(p+q) - \theta(p))} e^{\theta(p)x}. \end{aligned} \quad (16)$$

We now compute the integral on the positive half-line for each term. We have

$$\begin{aligned} & \int_0^\infty \left(\frac{\Phi(p+q) - \Phi(p)}{q} \right) \mathcal{H}^{(p+q, -q)}(x) \mathcal{H}^{(p, q)}(-y) dy \\ &= \frac{\Phi(p+q) - \Phi(p)}{q} \mathcal{H}^{(p+q, -q)}(x) \int_0^\infty e^{-\Phi(p)y} dy \\ &= \frac{\Phi(p+q) - \Phi(p)}{q\Phi(p)} \mathcal{H}^{(p+q, -q)}(x). \end{aligned} \quad (17)$$

At last, we notice that $x-y < x$ when $y > 0$ and then $\mathcal{W}_x^{(p, q)}(x-y) = \mathcal{W}^{(p)}(x-y)$. Consequently,

$$\begin{aligned} -\int_0^\infty \mathcal{W}_x^{(p, q)}(y) dy &= -\int_0^x \mathcal{W}^{(p)}(y) dy \\ &= -\frac{\alpha + \Phi(p)}{\Phi(p)\sqrt{\Delta_p}} e^{\Phi(p)x} + \frac{\alpha + \theta(p)}{\theta(p)\sqrt{\Delta_p}} e^{\theta(p)x} \\ &\quad - \frac{\alpha}{c\Phi(p)\theta(p)}. \end{aligned} \quad (18)$$

At last, summing up the three integrals in (16)–(18), we obtain

$$\begin{aligned} & \int_0^\infty e^{-pt} \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds} \right] dt \\ &= \frac{(\alpha + \theta(p)) \Phi(p+q)}{c(\Phi(p+q) - \theta(p))} \\ &\quad \times \left[\frac{1}{\Phi(p)\theta(p)} - \frac{1}{\Phi(p+q)\theta(p+q)} \right] e^{\theta(p)x} - \frac{\alpha}{c\Phi(p)\theta(p)}. \end{aligned} \quad (19)$$

The rest of the proof is similar to the proof of Proposition 3.2, i.e. we want to invert the double Laplace transform in (19). Fix $p > 0$. Since when $q \rightarrow \infty$ we have $\Phi(p+q) \rightarrow \infty$ and $\theta(p+q) \rightarrow -\alpha$, then the previous quantity converges to

$$\frac{\alpha + \theta(p)}{c\Phi(p)\theta(p)} e^{\theta(p)x} - \frac{\alpha}{c\Phi(p)\theta(p)} = -\frac{\alpha + \theta(p)}{\alpha p} e^{\theta(p)x} + \frac{1}{p}. \quad (20)$$

Since the distribution of $\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u) du$ has a mass at 0,

$$\mathbb{P}_x \left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u) du \in ds \right) = \alpha_t^x \delta_0(ds) + h_t^x(ds),$$

where $\alpha_t^x = \mathbb{P}_x(\tau_0^- > t) \in [0, 1]$ and $h_t^x(ds)$ is a nonnegative measure on $[0, t]$ with no mass at zero. From (20), we deduce that the Laplace transform of $t \mapsto \alpha_t^x$ is equal to $\frac{1}{p} - \frac{\alpha + \theta(p)}{\alpha p} e^{\theta(p)x}$ and then, from (19), the Laplace transform of $(s, t) \mapsto h_t^x(ds)$ is

$$\begin{aligned} \mathcal{L}h^x(p, q) &= \frac{(\alpha + \theta(p)) \Phi(p+q)}{c(\Phi(p+q) - \theta(p))} \\ &\quad \times \left[\frac{1}{\Phi(p)\theta(p)} - \frac{1}{\Phi(p+q)\theta(p+q)} \right] e^{\theta(p)x} \\ &\quad - \frac{\alpha + \theta(p)}{c\Phi(p)\theta(p)} e^{\theta(p)x}. \end{aligned}$$

Taking $x = 0$ in (19) and comparing with (8), we deduce that the Laplace transform of $(t, s) \mapsto h_t^x(ds)$ is equal to

$$\mathcal{L}h^x(p, q) = \left(\frac{c\Phi(p+q)}{p+q} - 1 \right) \frac{e^{\theta(p)x}}{c\Phi(p)}. \quad (21)$$

Therefore, to identify the law of $\int_0^t \mathbb{1}_{(-\infty, 0)}(X_u) du$, we need to identify

$$\alpha_t^x = \mathbb{P}_x(\tau_0^- > t) = 1 - \int_0^t \mathbb{P}_x(\tau_0^- \in ds).$$

Using the explicit expression for $\mathbb{P}_x(\tau_0^- \in ds)$ given in Drekić (2009, Equation (D.17)), we deduce that

$$\begin{aligned} \alpha_t^x &= 1 - \lambda e^{-\alpha x} \int_0^t e^{-(\lambda + c\alpha)s} \left[I_0 \left(2\sqrt{\lambda c\alpha} \sqrt{s(s+x/c)} \right) \right. \\ &\quad \left. - \frac{s}{s+x/c} I_2 \left(2\sqrt{\lambda c\alpha} \sqrt{s(s+x/c)} \right) \right] ds, \end{aligned}$$

where I_ν represents the modified Bessel function of the first kind of order ν , which can be written as

$$\begin{aligned} I_0(s) &= \frac{1}{\pi} \int_{-1}^1 e^{-su} (1-u^2)^{-1/2} du \quad \text{and} \\ I_2(s) &= \frac{s^2}{3\pi} \int_{-1}^1 e^{-su} (1-u^2)^{3/2} du. \end{aligned}$$

Moreover, using (13), the Laplace transform of h^x can be written in terms of the Laplace transform of α^x :

$$\mathcal{L}h^x(p, q) = \left[\frac{c\Phi(p+q)}{p+q} - 1 \right] \left[\mathcal{L}\alpha^x(p) + \frac{1}{p} (e^{\theta(p)x} - 1) \right].$$

We then deduce that (from the proof of Proposition 3.2)

$$h_t^x(ds) = c\alpha \left(a_s^0 + \frac{\lambda}{c\alpha} - 1 \right) (a_{t-s}^x + k_{t-s}^x) \mathbb{1}_{(0, t)}(s) ds,$$

with $\mathcal{L}k^x(p) = \frac{1}{p} (e^{\theta(p)x} - 1)$. As, for functions f and g , $\mathcal{L}f(p) = \frac{1}{p} \int_p^\infty \mathcal{L}g(q) dq$ implies $f(t) = \int_0^t s^{-1} g(s) ds$, we have $k_t^x = \int_0^t s^{-1} m_s^x ds + e^{-\alpha x} - 1$, with $\mathcal{L}m^x(p) = -x\theta'(p)e^{\theta(p)x}$. Let us now explicit the function m_t^x . We have

$$\begin{aligned} \mathcal{L}m^x(p) &= -\frac{x}{2c} \left(1 - \frac{p + \lambda + c\alpha}{\sqrt{(p + \lambda + c\alpha)^2 - 4c\alpha\lambda}} \right) \\ &\quad \times e^{\frac{x}{2c} \left(p + \lambda - c\alpha - \sqrt{(p + \lambda + c\alpha)^2 - 4c\alpha\lambda} \right)} \\ &= -\frac{xe^{-x\alpha}}{2c} \left(\sqrt{(p + \lambda + c\alpha)^2 - 4c\alpha\lambda} - (p + \lambda + c\alpha) \right) \\ &\quad \times \frac{e^{-\frac{x}{2c} \left(\sqrt{(p + \lambda + c\alpha)^2 - 4c\alpha\lambda} - (p + \lambda + c\alpha) \right)}}{\sqrt{(p + \lambda + c\alpha)^2 - 4c\alpha\lambda}}. \end{aligned}$$

Using similar inversion techniques as in Drekić (2009, (D.13)–(D.15)) the inverse Laplace transform of the previous expression can be expressed in terms of the modified Bessel function I_1 :

$$m_t^x = \frac{xe^{-x\alpha}}{2c} e^{-(\lambda + c\alpha)t} \frac{2\sqrt{c\alpha\lambda}t}{\sqrt{t(t+x/c)}} I_1 \left(2\sqrt{c\alpha\lambda} \sqrt{t(t+x/c)} \right).$$

Since $I_1(z) = \frac{z}{2} (I_0(z) - I_2(z))$, we have

$$\begin{aligned} m_t^x &= x\alpha\lambda e^{-x\alpha} e^{-(\lambda + c\alpha)t} \left[I_0 \left(2\sqrt{c\alpha\lambda} \sqrt{t(t+x/c)} \right) \right. \\ &\quad \left. - I_2 \left(2\sqrt{c\alpha\lambda} \sqrt{t(t+x/c)} \right) \right] t, \end{aligned}$$

and then

$$k_t^x = e^{-\alpha x} - 1 + \alpha \lambda e^{-\alpha x} \int_0^t e^{-(\lambda + c\alpha)s} \left[I_0 \left(2\sqrt{c\alpha\lambda} \sqrt{s(s+x/c)} \right) - I_2 \left(2\sqrt{c\alpha\lambda} \sqrt{s(s+x/c)} \right) \right] ds. \quad \square$$

If we set $x = 0$ in the definition of k_t^x in (12), then clearly $k_t^0 = 0$. Therefore, Theorem 2 is a generalization of Proposition 3.2.

3.3. Relationships with other definitions of ruin

When $r \rightarrow 0$, we should intuitively recover the classical probability of finite-time ruin. Indeed, when r is very small, then almost no grace period is granted when the surplus goes below zero.

First, for $x \geq 0$ and $t > 0$, we have for every $r > 0$ that

$$\left\{ \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds > r \right\} \subset \{ \tau_0^- \leq t \}$$

so

$$\mathbb{P}_x(\sigma_r \leq t) \leq \mathbb{P}_x(\tau_0^- \leq t).$$

Therefore, as a risk measure, the probability of cumulative Parisian ruin is less conservative than the probability of ruin.

We can obtain a much stronger relationship between cumulative Parisian ruin and classical ruin in the full generality of a Lévy insurance risk model. Recall that a Lévy insurance risk process is a process $X = \{X_t, t \geq 0\}$ with stationary and independent increments and no positive jumps.

Proposition 3.3. Assume X is a Lévy insurance risk process. For all $x \geq 0$, the time of cumulative Parisian ruin σ_r converges $\mathbb{P}_x$ -almost surely, as $r \rightarrow 0$, to the time of classical ruin τ_0^- .

Proof. First, we show convergence in probability (with respect to $\mathbb{P}_x$, for an arbitrary value of $x \geq 0$), that is, for every $\epsilon > 0$,

$$\mathbb{P}_x(|\sigma_r - \tau_0^-| > \epsilon) \xrightarrow{r \rightarrow 0} 0.$$

Indeed, for any $\epsilon > 0$, we note that $\sigma_r \geq \tau_0^-$ and, by the strong Markov property,

$$\begin{aligned} \mathbb{P}_x(|\sigma_r - \tau_0^-| > \epsilon) &= \mathbb{P}_x\left(\int_{\tau_0^-}^{\epsilon + \tau_0^-} \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq r\right) \\ &= \mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left(\int_0^\epsilon \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq r\right)\right]. \end{aligned}$$

Therefore we want to show that $\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left(\int_0^\epsilon \mathbb{1}_{(-\infty, 0)}(X_s) ds = 0\right)\right] = 0$. Clearly, on $\{X_{\tau_0^-} < 0\}$, since the process is skip-free upward,

$$\mathbb{P}_{X_{\tau_0^-}}\left(\int_0^\epsilon \mathbb{1}_{(-\infty, 0)}(X_s) ds = 0\right) = 0.$$

On the other hand, if $x > 0$, it is well known that X crosses zero continuously if and only if it has a Brownian component. Finally, recall that a SNLP has paths of unbounded variation (e.g. with a Brownian component) if and only if 0 is regular for $(-\infty, 0)$, which means that when the process sits at zero it will enter $(-\infty, 0)$ immediately (and will start to accumulate occupation time). Putting the pieces together concludes the proof of convergence in probability.

Since $r \mapsto \sigma_r$ is a non-decreasing function, we deduce that σ_r converges to τ_0^- a.s. when r goes to 0. $\square$

At first sight, it is very tempting to think of exponential Parisian ruin as being *one Laplace transform away* from deterministic Parisian ruin. As discussed in Loeffen et al. (2013), due to the choice of different exponential clocks for different excursions below zero, this is not the case. In fact, it turns out that cumulative Parisian ruin is the notion of Parisian ruin that can be connected to exponential Parisian ruin through a Laplace transform, as the next result will show.

Note that the next result holds for any Lévy insurance risk process, including of course the Cramér–Lundberg risk process and the Brownian risk process studied previously.

Proposition 3.4. Assume X is a Lévy insurance risk process and let $\mathbf{e}_q$ be an independent exponential random variable with rate $q > 0$. Then, for all initial surplus $x \geq 0$, κ_q and $\sigma_{\mathbf{e}_q}$ are such that, for all $x \geq 0$, $t > 0$,

$$\mathbb{P}_x(\kappa_q \leq t) = \mathbb{P}_x(\sigma_{\mathbf{e}_q} \leq t)$$

and, moreover,

$$\mathbb{P}_x(\kappa_q < \infty) = \mathbb{P}_x(\sigma_{\mathbf{e}_q} < \infty).$$

Proof. Comparing Guérin and Renaud (2016, Corollary 1) and Baurdoux et al. (2016, Corollary 1.1), we observe that, for all $y \in (-\infty, 0]$,

$$\begin{aligned} q \int_0^\infty e^{-\theta t} \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds}, X_t \in dy \right] dt \\ = \mathbb{E}_x \left[e^{-\theta \kappa_q}, X_{\kappa_q} \in dy, \kappa_q < \infty \right]. \end{aligned}$$

Using Fubini's theorem and then integrating over $(-\infty, 0]$ (with respect to y) both distributions, we get

$$\begin{aligned} q \int_0^\infty e^{-\theta t} \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds}, X_t < 0 \right] dt \\ = \mathbb{E}_x \left[e^{-\theta \kappa_q}, \kappa_q < \infty \right]. \end{aligned}$$

Note that we can write

$$\mathbb{E}_x \left[e^{-\theta \kappa_q}, \kappa_q < \infty \right] = \int_0^\infty e^{-\theta t} \mathbb{P}_x(\kappa_q \in dt).$$

Therefore, by Laplace inversion, we have, on $(0, \infty)$,

$$q \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds}, X_t < 0 \right] dt = \mathbb{P}_x(\kappa_q \in dt).$$

Integrating over $(0, t)$ for an arbitrary value of t yields

$$\begin{aligned} \mathbb{P}_x(\kappa_q \leq t) &= \int_0^t q \mathbb{E}_x \left[e^{-q \int_0^s \mathbb{1}_{(-\infty, 0)}(X_u) du}, X_s < 0 \right] ds \\ &= \mathbb{E}_x \left[\int_0^t q \mathbb{1}_{(-\infty, 0)}(X_s) e^{-q \int_0^s \mathbb{1}_{(-\infty, 0)}(X_u) du} ds \right] \\ &= \mathbb{E}_x \left[1 - e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds} \right] \\ &= 1 - \mathbb{E}_x \left[e^{-q \int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds} \right]. \end{aligned}$$

So, we have

$$\begin{aligned} \mathbb{P}_x(\kappa_q \leq t) &= 1 - \mathbb{P}_x \left(\int_0^t \mathbb{1}_{(-\infty, 0)}(X_s) ds \leq \mathbf{e}_q \right) \\ &= \mathbb{P}_x(\sigma_{\mathbf{e}_q} \leq t), \end{aligned} \quad (22)$$

and, in particular,

$$\begin{aligned} \mathbb{P}_x(\kappa_q < \infty) &= 1 - \mathbb{E}_x \left[e^{-q \int_0^\infty \mathbb{1}_{(-\infty, 0)}(X_s) ds} \right] \\ &= \mathbb{P}_x(\sigma_{\mathbf{e}_q} < \infty). \quad \square \end{aligned} \quad (23)$$

Remark 3.1. The previous proof highlights a strong relationship between the distribution of the occupation-time process and exponential Parisian ruin, namely identity (22). The equality in Eq. (23) was first discussed in Landriault et al. (2011); note that Eq. (22) is a finite-time version of it.

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References

- Akahori, J., 1995. Some formulae for a new type of path-dependent option. *Ann. Appl. Probab.* 5 (2), 383–388.
- Albrecher, H., Cheung, E.C. K., Thonhauser, S., 2011. Randomized observation periods for the compound Poisson risk model: dividends. *Astin Bull.* 41 (2), 645–672.
- Albrecher, H., Cheung, E.C. K., Thonhauser, S., 2013. Randomized observation periods for the compound Poisson risk model: the discounted penalty function. *Scand. Actuar. J.* 6, 424–452.
- Albrecher, H., Gerber, H.U., Shiu, E.S. W., 2011. The optimal dividend barrier in the Gamma-Omega model. *Eur. Actuar. J.* 1 (1), 43–55.
- Albrecher, H., Lautscham, V., 2013. From ruin to bankruptcy for compound Poisson surplus processes. *Astin Bull.* 43 (2), 213–243.
- Asmussen, S., Albrecher, H., 2010. *Ruin Probabilities*, second ed. World Scientific.
- Baurdoux, E.J., Pardo, J.C., Pérez, J.L., Renaud, J.-F., 2016. Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes. *J. Appl. Probab.* 53 (2), 572–584.
- Czarna, I., Palmowski, Z., 2011. Ruin probability with Parisian delay for a spectrally negative Lévy risk process. *J. Appl. Probab.* 48 (4), 984–1002.
- dos Reis, A.E., 1993. How long is the surplus below zero? *Insurance Math. Econom.* 12 (1), 23–38.
- Drekic, S., 2009. Discussion of “On the joint distributions of the time to ruin, the surplus prior to ruin, and the deficit at ruin in the classical risk model, by D. Landriault and G.E. Willmot, vol. 13, no. 2, 2009”. *N. Am. Actuar. J.* 13 (3), 404–406.
- Gerber, H.U., Shiu, E.S.W., 1998. On the time value of ruin. *N. Am. Actuar. J.* 2 (1), 48–78.
- Gerber, H.U., Shiu, E.S. W., Yang, H., 2012. The Omega model: from bankruptcy to occupation times in the red. *Eur. Actuar. J.* 2 (2), 259–272.
- Guérin, H., Renaud, J.-F., 2016. Joint distribution of a spectrally negative Lévy process and its occupation time, with step option pricing in view. *Adv. Appl. Probab.* 48 (1), 274–297.
- Hugonnier, J.-N., 1999. The Feynman-Kac formula and pricing occupation time derivatives. *Int. J. Theor. Appl. Finance* 2 (2), 153–178.
- Karatzas, I., Shreve, S.E., 1991. *Brownian Motion and Stochastic Calculus*, second ed. Springer-Verlag, New York.
- Kyprianou, A.E., 2013. Gerber-Shiu Risk Theory. In: *EAA Series*, Springer.
- Kyprianou, A.E., 2014. Fluctuations of Lévy processes with applications - Introductory lectures, second ed. In: *Universitext*, Springer, Heidelberg.
- Landriault, D., Renaud, J.-F., Zhou, X., 2011. Occupation times of spectrally negative Lévy processes with applications. *Stochastic Process. Appl.* 121 (11), 2629–2641.
- Landriault, D., Renaud, J.-F., Zhou, X., 2014. An insurance risk model with Parisian implementation delays. *Methodol. Comput. Appl. Probab.* 16 (3), 583–607.
- Loeffen, R.L., Czarna, I., Palmowski, Z., 2013. Parisian ruin probability for spectrally negative Lévy processes. *Bernoulli* 19 (2), 599–609.
- Loeffen, R.L., Renaud, J.-F., Zhou, X., 2014. Occupation times of intervals until first passage times for spectrally negative Lévy processes. *Stochastic Process. Appl.* 124 (3), 1408–1435.
- Loisel, S., Trufin, J., 2014. Properties of a risk measure derived from the expected area in red. *Insurance Math. Econom.* 55, 191–199.
- Makarov, R., Metzler, A., Ni, Z., 2015. Modelling default risk with occupation times. *Financ. Res. Lett.* 13, 54–65.
- Mitric, I.-R., Trufin, J., 2016. On a risk measure inspired from the ruin probability and the expected deficit at ruin. *Scand. Actuar. J.* (10), 932–951.
- Polyanin, A.D., Manzhirov, A.V., 2008. *Handbook of Integral Equations*, second ed. Chapman & Hall/CRC, Boca Raton, FL.
- Takács, L., 1996. On a generalization of the arc-sine law. *Ann. Appl. Probab.* 6 (3), 1035–1040.
- Trufin, J., Albrecher, H., Denuit, M., 2011. Properties of a risk measure derived from ruin theory. *Geneva Risk Insur. Rev.* 36, 174–188.
- Yildirim, Y., 2006. Modelling default risk: a new structural approach. *Financ. Res. Lett.* 3 (3), 165–172.