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ORIGINAL RESEARCH PAPER

On the depletion problem for an insurance risk process: new non-ruin quantities in collective risk theory

Zied Ben-Salah[1] · Hélène Guérin[2] · Manuel Morales[3] · Hassan Omidi Firouzi[3]

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Abstract The field of risk theory has traditionally focused on ruin-related quantities. In particular, the so-called expected discounted penalty function (Gerber and Shiu. N Am Actuar J, 2(1):48–78, 1998) has been the object of a thorough study over the years. Although interesting in their own right, ruin related quantities do not seem to capture path-dependent properties of the reserve. In this article we aim at presenting the probabilistic properties of drawdowns and the speed at which an insurance reserve depletes as a consequence of the risk exposure of the company. These new quantities are not ruin related yet they capture important features of an insurance position and we believe it can lead to the design of a meaningful risk measures. Studying drawdowns and speed of depletion for Lévy insurance risk

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✉ Manuel Morales
morales@dms.umontreal.ca

Zied Ben-Salah
zied.bensalah@aucegypt.edu

Hélène Guérin
helene.guerin@univ-rennes1.fr

Hassan Omidi Firouzi
homidifirouzi@escpeurope.eu

[1] Department of Mathematics and Actuarial Science, The American University in Cairo, PO Box 74, New Cairo 11835, Cairo, Canada

[2] IRMAR. UMR CNRS 6625., University of Rennes 1, Campus de Beaulieu, 35042 Rennes Cedex, France

[3] Laboratory of Excellence for Financial Regulation (LABEX ReFi), Université Paris 1, 81 avenue de la Republique, Paris 75543 Paris cedex 11, France

processes represent a novel and challenging concept in insurance mathematics. In this paper, all these concepts are formally introduced in an insurance setting. Moreover, we specialize recent results in fluctuation theory for Lévy processes (Mijatovic and Pistorius. StochProcess Their Appl, 22:3812–3836, 2012) in order to derive expressions for the distribution of several quantities related to the depletion problem. Of particular interest are the distribution of drawdowns and the Laplace transform for the speed of depletion. These expressions are given for several examples of Lévy insurance risk processes for which they can be calculated, in particular for the classical Cramér–Lundberg model.

1 Introduction

Traditionally, collective risk theory is mainly concerned with the ruin problem which is nicely encapsulated in the concept of expected discounted penalty function (EDPF) introduced in [10]. This so-called Gerber-Shiu function is a functional of the ruin time (i.e., the first time the reserve level of a firm becomes negative), the surplus prior to ruin, and the deficit at ruin. The EDPF has been extensively studied and generalized to various scenarios and there is now a wide range of models for which expressions of the EDPF are available. All of these models incorporate different levels of complexity into the picture.

In particular, the so-called Lévy insurance risk processes have been the object of much attention in the last decade, mainly because they nicely generalize the Cramér–Lundberg model while allowing to bring new insight into the field of ruin theory through the well-developed theory of fluctuations for such processes. Several families of Lévy processes have been put forward as risk models and we now have a well-established literature on the subject. For a thorough discussion on the suitability of these processes as risk models we refer the reader to [9, 21] and references therein.

As it turns out, the first-passage problem for Lévy processes is well understood and recent results in this area have been applied to the ruin problem in order to gain interesting insight (see for instance [3, 4, 16]). In this paper, we focus yet again on Lévy insurance risk processes because of the extensive set of tools available for this family of stochastic processes. Through concepts originally developed for the study of the first-passage time problem, we can now study questions that go beyond the ruin problem and that are connected to path-properties of the process that give a tell-telling picture of how depletion occurs.

Quantities such as the speed of depletion and drawdowns have been studied in finance in connection to the concept of market crash [25]. Indeed, in finance one would be interested in knowing how fast and how frequent drawdowns of a certain size occur. In insurance, these questions have not been studied yet, despite the fact that these concepts are meaningful from an insurance risk management point of view. Clearly knowing how your insurance reserve is affected by drawdowns and how fast and frequent these are could be useful to devise risk management tools. These quantities provide a measure of riskiness that is not linked to the ruin event but rather to the depletion features of the reserve. However, this problem is technically challenging due to the jump nature of insurance models.

The aim of this paper is threefold. One one hand, we aim at introducing the problem of depletion into the theory of collective risk theory as a meaningful question from a risk management point of view. We formally define new non-ruin quantities within the classical risk theory framework and we discuss their main features and advantages over traditional ruin-related quantities. Indeed, it is interesting to notice that all of the available research focuses on ruin-related quantities which, by their very nature, fail to explain how an insurance reserve depletes over time. Thus, although ruin theory provides a good probabilistic picture of the problem of insolvency of an insurance reserve, it cannot explain other features that are equally representative of the riskiness of an insurance reserve such as its speed of depletion and the frequency of drawdowns. The ruin event is an object of concern over the long-run but a risk manager might also keep an eye on any series of particularly large drawdowns especially if they happen particularly fast. So concerning oneself with the ruin event overlooks other risky events that also have an impact on the solvency and financial planning of an insurance company.

A second objective is to actually derive expressions for the distribution of several depletion-related random variables. As it turns out, recent results in the theory of fluctuations for one-sided Lévy processes [20] can be specialized in order to derive expressions for these depletion-related quantities. Key to the derivation of such expressions is the scale function of the process driving the insurance risk model. As we discuss, general and non-explicit expressions for the distribution of random variables in the depletion problem can only be simplified if a simple form for the scale function is available.

Hence, a third objective is to study several examples of insurance risk processes in the literature for which q-scale functions are explicit or semi-explicit. For such examples, we derive explicit or semi-explicit expressions for the distribution of depletion-related quantities. In particular, we study exhaustively the case of a classical Cramér-Lundberg model driven by a compound Poisson process with exponential jumps. Not surprisingly, this simple case has always yielded text-book examples of closed-form solutions in the risk theory literature. The problem of depletion is no exception and we present explicit expressions for the distribution of several depletion-related random variables in this case. We also provide a similar analysis for other examples of insurance risk models possessing simple scale functions, namely the gamma subordinator, the stable family of processes, the Brownian motion with drift and the meromorphic *beta*-process.

This paper is organized as follows. In Sect. 2 we introduce a general model based on a Lévy risk process for which we define the depletion problem and the notions of *drawdowns* and *speed of depletion* as well as related variables. Some preliminary results from the theory of fluctuations for Lévy processes are given in Sect. 3. In Sect. 4 we study the problem of depletion for an insurance Lévy risk process and we give general expressions for the distribution of depletion random variables of interest. In Sect. 5, we derive explicit expressions for all depletion-related quantities for several examples of Lévy insurance risk processes. Finally, in Sect. 6, we provide a simulation study for some of the risk models discussed in Sect. 5.

2 The depletion problem for an insurance risk model

We consider a very general setup that generalizes the standard Cramér-Lundberg model. We consider in this paper an insurance risk process $X = (X_t)_{t \geqslant 0}$ with X a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geqslant 0}, \mathbb{P})$ and $X_0 = 0 \ (\mathbb{P} - a.s)$. For technical reasons that will become clear later in the paper, we restrict ourselves to those processes having paths of unbounded variation or paths of bounded variation as well as a Lévy measure which is absolutely continuous with respect to the Lebesgue measure. In order to avoid the case of trivial reflected processes we exclude processes with monotone paths. As is customary, the symbols $\mathbb{E}_x$ and $\mathbb{P}_x$ will denote the expectation and the probability measure related to the process started at x, i.e. the expectation and law of the process $x + X$ under $\mathbb{P}$ and if the process is started from zero we will use simple notations $\mathbb{E}$ and $\mathbb{P}$.

Notice that a such model contains all elements of a traditional risk model and encompasses, among others, the risk models studied in [4, 8, 12, 21]. Indeed, the constant rate premium is included as the drift of X, the so-called perturbation comes in as the Brownian component of X and the pure aggregate claims is present as the jump part of X, which could be set as a compound Poisson or an infinite activity process. With this in mind, we assume the process X to have a positive drift such that $\mathbb{E}[X_1] > 0$. Notice that in traditional ruin theory, this assumption responds to both, technical and practical reasons. Technically, it is needed in order to avoid the possibility that X becomes negative almost surely whereas from a practical point of view it makes sense since it is common practice in insurance to work with loaded premiums. Indeed, it is standard to write the drift component within X in terms of a safety loading. For instance, notice that we can recuperate the classical Cramér–Lundberg model if $X_t = c\,t - S_t$ where $c := (1 + \theta)\mathbb{E}[S_1]$ and S is a compound Poisson process modeling aggregate claims. The drift c, with a positive safety loading $\theta > 0$, is the collected premium rate. In the context of the depletion problem we do not need this condition. We keep it here for purely practical reasons as it is common practice to have insurance loaded premiums.

One of the advantages of considering a general Lévy risk model is that we can use the tools and methods of the fluctuation theory of Lévy processes, allowing for a somewhat deeper understanding of the ruin problem but also of the depletion problem which can prove to yield just as interesting information about the riskiness of the reserve.

For a more extensive discussion on Lévy risk models we refer to [9]. In this paper we will specialize this setting to five examples of Lévy processes that have been studied in the literature in the context of the ruin problem.

One of the main objects of interest in ruin theory is the *ruin time*, τ, representing the first passage time of an insurance Lévy risk process X below zero when $X_0 = x$, i.e.

$$\tau := \inf\{t > 0 \ : \ X_t < 0\}, \tag{1}$$

where we set $\tau = +\infty$ if $X_t \geq 0$ for all $t \geqslant 0$.

In this paper, our main object of concern is the *depletion problem*. Throughout this paper, we use the term *depletion problem* to refer to the analysis of several random quantities related to how the risk reserve is depleted. This notion has two distinct random times as its main building blocks. In the following we give a mathematical description of this problem starting with some preliminary notation.

We define the running infimum and the running supremum of a given Lévy process X respectively by

$$\underline{X}_t := \inf_{0 \leqslant s \leqslant t} X_s \qquad \text{and} \qquad \overline{X}_t := \sup_{0 \leqslant s \leqslant t} X_s \ .$$

Now we characterize the depletion problem for X. We first define the *drawdown* process $Y = (Y_t)_{t \geqslant 0}$, associated with a given risk process X, to be

$$Y_t := \overline{X}_t - X_t \ , \qquad t \geqslant 0 \ . \tag{2}$$

The first-passage time over a level $a > 0$ of the drawdown process Y is then defined to be

$$\tau_a := \inf\{t \geqslant 0 : Y_t > a\} \ . \tag{3}$$

It is well-known that $\tau_a < \infty$ $\mathbb{P}$-almost surely (see [1], Theorem 1). Just like the ruin time in (1), this new random time in (3) contains relevant information on potentially risky behavior of the reserve. Their distribution can be used to measure the likeliness of path-related events that might have a negative impact on the financial health of the reserve. The random time τ_a records the time at which a drawdown in the reserve is larger than a, previously agreed upon, critical level a. An interesting set of associated tale-telling random variables can be built upon the random time (3). First, we need to define a process that will be useful in constructing meaningful non-ruin quantities. The last time before t that X reaches its running supremum, denoted by $\overline{G}_t$, is defined as

$$\overline{G}_t := \sup\{s \leq t : X_s \quad \text{or} \quad \mathrm{X_{s-}} = \overline{\mathrm{X}}_{\mathrm{s}}\}. \tag{4}$$

Thus the time τ_a of the critical drawdown of size a along with the following quantities characterize the depletion problem for X:

- the last time the reserve was at its maximum level prior to critical drawdown, $\overline{G}_{\tau_a}$;
- the *speed of depletion*, $\tau_a - \overline{G}_{\tau_a}$;
- the maximum reserve level attained before critical drawdown is observed, $\overline{X}_{\tau_a}$;
- the minimum reserve level prior to critical drawdown, $\underline{X}_{\tau_a}$;
- the largest drawdown observed before critical drawdown of size a, $Y_{\tau_a -}$;
- the overshoot of the critical drawdown over level a, $Y_{\tau_a} - a$.

Clearly, these variables contain information on the how the insurance deserve depletes over time. All of these quantities encapsulate relevant knowledge about the critical drawdown event (Fig. 1). A risk manager would be potentially interested in

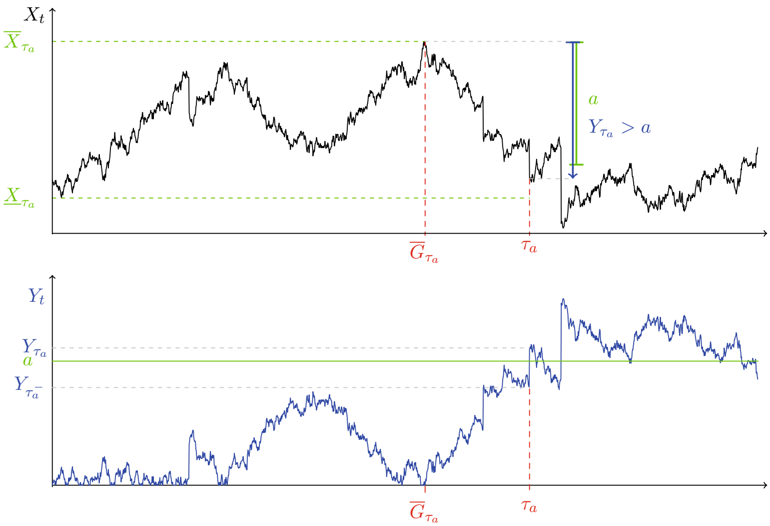


Fig. 1 A path of $X_t = 10 + t + 2B_t - S_t$, the corresponding drawdown process Y, and their related depletion quantities, where $(B_t)_{t \geq 0}$ is a standard Brownian motion and S is an independent compound Poisson process with Lévy measure $\nu(dx) = e^{-2y}dx$

gaining information regarding the distribution of the time of the critical drawdown of size a, i.e. $\mathbb{P}_x(\tau_a \leqslant t)$. This gives information on how likely the reserve is to face a critical drawdown within a given time interval. Even more valuable information can be found in the distribution of the *speed of depletion*, this random variable indicates how fast critical drawdowns tend to occur. It should be noticed that this is not an actual *speed* but rather a length of time. We decided in favor of this particular choice of words in this context since it is the same notion used in finance to describe the *speed of market crash* [see 25]. A drawdown is not as alarming if it happens over a long period than if it happens suddenly. Information about the distributions of the maximum and minimum reserve levels prior to critical drawdown and of the largest drawdown on record before critical drawdown sheds light on the structure of the depletion event. It is also interesting to know how large (or not) critical drawdowns tend to be, that is, by how much they overshoot the critical level a when they occur. In fact, the level a itself can be set by using the distribution of the overshoot. Since this distribution is a function of a we can decide what a critical drawdown size is depending on how likely certain levels are. In Fig. 1 we show a simulated path for an insurance risk process in order to illustrate these quantities.

It is interesting to notice that there is a connection between ruin and depletion through the distribution of the minimum reserve level prior to critical drawdown. We will see that, if expressions are available, we can calculate the probability that ruin occurs before a critical drawdown of size a.

In general, just like in the ruin problem, knowledge on the probabilistic properties of such quantities could be relevant in risk management applications. Although critical drawdowns do not spell immediate doom for the company as ruin

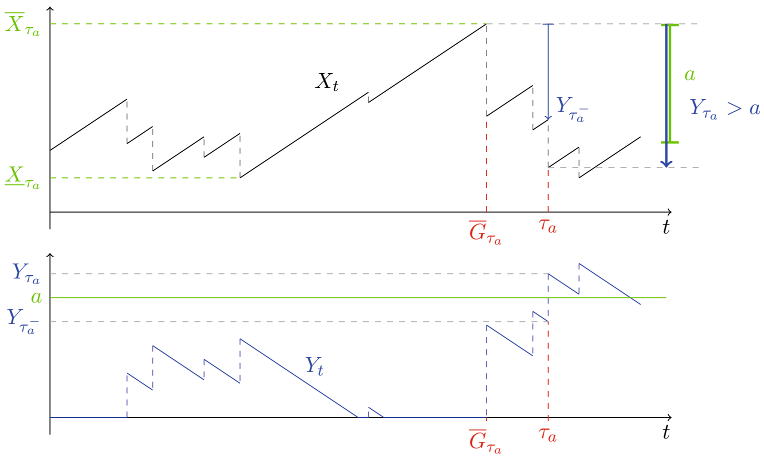


Fig. 2 A path of a compound Poisson process X, the corresponding drawdown process Y and their related depletion quantities

does, a large enough drawdown might be a warning sign that a risk manager might want to take into account. This information coupled with knowledge on how fast these critical drawdowns happen could be used to design risk measures and or management policies that will ensure the solvency of the reserve.

Once that these non-ruin quantities have been introduced, the aim of the paper is to derive expressions for the probability measure of these random variables associated with the depletion problem. This will be done in detail for five examples of Lévy insurance risk processes. But before we need to introduce some preliminary results that are key to our analysis.

3 Drawdowns for spectrally negative lévy processes

In this section, we introduce some notions and results that are needed in the rest of the paper. Let $X = (X_t)_{t\geqslant 0}$ be a spectrally negative Lévy process defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, \mathbb{P})$. We impose the same restrictions as in [1], i.e. X has either paths of unbounded variation or paths of bounded variation as well as a Lévy measure which is absolutely continuous with respect to the Lebesgue measure. Further, we exclude processes with monotone paths.

Since X has no positive jumps, the expectation $\mathbb{E}[e^{sX_t}]$ exists for all $s \geqslant 0$ and it is given by $\mathbb{E}[e^{sX_t}] = e^{t\psi(s)}$ where $\psi(s)$ is of the form

$$\psi(s) = d\,s + \frac{1}{2}\sigma^2 s^2 + \int_0^\infty (e^{-x\,s} - 1 + s\,x\mathbb{1}_{\{x<1\}})\,\nu(dx)\,, \tag{5}$$

where $d \in \mathbb{R}$, $\sigma > 0$ and ν is the Lévy measure associated with the process $-X$ (for a thorough account on Lévy process see [2, 17]).

We introduce Φ the right inverse of ψ, defined on $[0, \infty)$ by

$$\Phi(q) := \sup\{s \geqslant 0 : \psi(s) = q\}\ . \tag{6}$$

Notice that since X is a spectrally negative Lévy process X, we have that $\Phi(q) > 0$ for $q > 0$ (see [17]).

It is well-known that, for every $q \geqslant 0$, there exists a function $W^{(q)} : \mathbb{R} \longrightarrow [0, \infty)$ such that $W^{(q)}(y) = 0$ for all $y<0$ satisfying

$$\int_0^{\infty} e^{-\lambda y} W^{(q)}(y)dy = \frac{1}{\psi(\lambda) - q}, \qquad \lambda > \Phi(q). \tag{7}$$

This is the so-called *q-scale functions* $\{W^{(q)},\ q \geqslant 0\}$ of the process X (see [17]) and it is a key notion in the analysis of drawdowns for spectrally negative Lévy processes. Notice that for $q = 0$, Eq. (7) defines the so-called scale function and we simply write W.

Before discussing drawdowns, we need to introduce additional functions related to the q-scale function. Let $W_+^{\prime(q)}$ be the right derivative function of the q-scale. Following the notation in [20], we denote the ratio of the right derivative of the q-scale function and the q-scale function at $a > 0$ by

$$\lambda(a, q) := \frac{W_+^{\prime(q)}(a)}{W^{(q)}(a)}\ . \tag{8}$$

We can now define, for any $a > 0$ and $p, q > 0$, the mapping $F_{p,q,a} : \mathbb{R}_+ \to \mathbb{R}_+$

$$F_{p,q,a}(y) := \lambda(a, q)e^{-y\lambda(a,p)}\ . \tag{9}$$

Moreover, consider the q-resolvent measure $R_a^{(q)}(dy) = \mathbb{E}\left[\int_0^{\tau_a} e^{-qt}\mathbb{1}_{Y_t \in dy}dt\right]$ of Y killed upon first exit from [0, a] which can be expressed in the following way (see [22], Theorem 1),

$$R_a^{(q)}(dy) := \left[\lambda(a, q)^{-1}W^{(q)}(dy) - W^{(q)}(y)dy\right], \qquad y \in [0, a]\ , \tag{10}$$

and the function

$$\Delta^{(q)}(a) = \frac{\sigma^2}{2}\left[W^{\prime(q)}(a) - \lambda(a, q)^{-1}W^{\prime\prime(q)}(a)\right] \tag{11}$$

with $\Delta^{(q)}(a) = 0$ when $\sigma = 0$.

The functions in Eqs. (8), (9), (10) and (11) will frequently appear throughout the paper. The following theorem will play a key role in our contribution. For a thorough discussion and a proof, we refer to [20].

Theorem 1 *Consider a spectrally negative Lévy process X such that $X_0 = x \in \mathbb{R}$, with paths of unbounded variation or with a Lévy measure which is absolutely continuous with respect to the Lebesgue measure. Let further Y be its associated drawdown process defined in* (2). *Let τ_a be the stopping time in* (3) *so we can define the following events, for a given $a > 0$,*

$$A_0 = \{\underline{X}_{\tau_a} \geqslant u, \overline{X}_{\tau_a} \in dv, Y_{\tau_a-} \in dy, Y_{\tau_a} - a \in dh\} \quad \text{and}$$
$$A_c = \{\underline{X}_{\tau_a} \geqslant u, \overline{X}_{\tau_a} \in dv, Y_{\tau_a} = a\}, \tag{12}$$

where u, v, y *and* h *satisfy*

$$u \leq x, y \in [0, a], v \geqslant x \vee (u + a) \quad \text{and} \quad \mathrm{h} \in (0, \mathrm{v} - \mathrm{u} - \mathrm{a}].$$

Then, for any $q, r \geqslant 0$ *the following identities hold true*:

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}} \mathbb{1}_{A_0}\right] = \frac{W^{(q+r)}((x-u) \wedge a)}{W^{(q+r)}(a)} F_{q+r,q,a}(v - (x \vee (u+a)))R_a^{(q)}(dy)\nu(a - y + dh)dv, \tag{13}$$

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}} \mathbb{1}_{A_c}\right] = \frac{W^{(q+r)}((x-u) \wedge a)}{W^{(q+r)}(a)} F_{q+r,q,a}(v - (x \vee (u+a)))\Delta^{(q)}(a)dv, \tag{14}$$

where $\mathbb{1}$ *is the standard indicator function*, ν *is the Lévy measure of* X *that appears in* (5), $x \vee y = \max(x, y)$, $x \wedge y = \min(x, y)$ *and* $\overline{G}_t$ *is the process defined in Eq.* (4).

The assumptions of Theorem 1 are in no way restrictive since most of the risk insurance processes in the literature fall within this class of processes, i.e., they are defined through a Lévy density.

We also remark that on the event A_0 defined in Eq. (12), the critical drawdown is performed by a jump of the Lévy process X while it is performed continuously on the event A_c. These two events and the expectations in Eqs. (13) and (14) in Theorem 1 contain all information regarding the depletion problem. The aim of this paper is to provide explicit expressions for the distribution of these depletion-related random variables under relevant insurance models.

4 Analysis of the depletion problem

The main goal of this paper is to study the depletion event as told by the quantities in Theorem 1. In principle, we can study through the expectations in Eqs. (13) and (14) the probability measure of all quantities involved as well as the Laplace transform of the speed of depletion. This can be accomplished by setting $q = r = 0$ and/or integrating over a suitable set those expressions in Eqs. (13) and (14). How explicit these expressions are will depend on the form of the q-scale function and the Lévy measure of the model. Nonetheless, in this section we give some general results that bring insight into the problem.

It turns out that there is a link between the running infimum at time τ_a given by (3) and the ruin time τ given by (1). We can easily deduce that $\tau_a \leq \tau$ a.s. on the event $\{\underline{X}_{\tau_a} \geqslant 0\}$, while $\{\tau_a < \tau\}$ implies $\{\underline{X}_{\tau_a} \geqslant 0\}$.

Furthermore, from the definition of these quantities we can see that when the initial surplus x is strictly greater than a, then $\underline{X}_{\tau_a-} > 0$ and the hitting time of the critical drawdown is smaller than the ruin time, i.e. $\tau_a \leq \tau$ a.s. On the other hand, when $x < a$, ruin can occur before the critical drawdown.

We can now state a result which makes a link between the ruin event and the depletion problem.

Theorem 2 *Consider an insurance risk process $(X_t)_{t \geqslant 0}$ with initial surplus $x \geqslant 0$ satisfying assumptions of Sect. 2 and let $a > 0$be a fixed critical drawdown size. Then,*

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \frac{W(x \wedge a)}{W(a)} + \frac{W(x \wedge a)}{W(a)} \int_{y \in [0,a]} \int_{h > 0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh) R_a^{(0)}(dy), \tag{15}$$

where W is the scale function, ν the Lévy measure of X and $R_a^{(0)}$ is defined by Eq. (10) with $q = 0$.

Proof We notice that $\mathbb{P}_x\left(\underline{X}_{\tau_a} < 0\right) = 1 - \mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0)$ and

$$\mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0) = \mathbb{P}_x\left(\underline{X}_{\tau_a} \geqslant 0,\ Y_{\tau_a} > a\right) + \mathbb{P}_x\left(\underline{X}_{\tau_a} \geqslant 0,\ Y_{\tau_a} = a\right).$$

Then putting $q = r = u = 0$ and integrating (13) and (14) with respect to $v \in [x \vee a, \infty)$, $y \in [0, a]$ and $h \in (0, v - a]$, we have

$$\mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0) = \frac{W(x \wedge a)}{W(a)} \left[\int_{y \in [0,a]} \left(\int_{v \geqslant x \vee a} F_{0,0,a}(v - x \vee a) \int_{h \in (0, v-a]} \nu(a - y + dh) dv \right) R_a^{(0)}(dy) + \int_{v \geqslant x \vee a} F_{0,0,a}(v - x \vee a) \Delta^{(0)}(a) dv \right].$$

Since $F_{0,0,a}$ is defined by (9), using Fubini's theorem, the previous expression gives

$$\mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0) = \frac{W(x \wedge a)}{W(a)} \left[e^{\lambda(a,0) x \vee a} \int_{y \in [0,a]} \int_{h > 0} e^{-\lambda(a,0)(x \vee (h+a))} \nu(a - y + dh) R_a^{(0)}(dy) + \Delta^{(0)}(a) \right]. \tag{16}$$

We notice that taking $u \to -\infty$ and integrating (13) and (14) with respect to $v \in [x \vee a, \infty)$, $h \in (0, v - a]$ and $y \in [0, a]$ with $r = q = 0$,

$$\int_0^a R_a^{(0)}(dy) \int_0^\infty \nu(a - y + dh) + \Delta^{(0)}(a) = 1. \tag{17}$$

Using this remark, we deduce that (16) gives for $x \leq a$

$$\begin{aligned}\mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0) &= \frac{W(x)}{W(a)}\left[\int_{y\in[0,a]}\int_{h>0} e^{-\lambda(a,0)h} v(a-y+dh)R_a^{(0)}(dy) + \Delta^{(0)}(a)\right]\\ &= \frac{W(x)}{W(a)}\left[1-\int_{y\in[0,a]}\int_{h>0}\left(1-e^{-\lambda(a,0)h}\right)v(a-y+dh)R_a^{(0)}(dy))\right],\end{aligned}$$

and for $x > a$,

$$\begin{aligned}&\mathbb{P}_x(\underline{X}_{\tau_a} \geqslant 0)\\ &= \int_{y\in[0,a]}\int_0^{x-a} v(a-y+dh)R_a^{(0)}(dy)\\ &\quad + \int_{y\in[0,a]}\int_{x-a}^{\infty} e^{-\lambda(a,0)(h+a-x)} v(a-y+dh)R_a^{(0)}(dy) + \Delta^{(0)}(a)\\ &= 1-\int_{y\in[0,a]}\int_{x-a}^{\infty}\left(1-e^{-\lambda(a,0)(h+a-x)}\right)v(a-y+dh)R_a^{(0)}(dy)\\ &= 1-\int_{y\in[0,a]}\int_{h>0}\left(1-e^{-\lambda(a,0)h}\right)v(x-y+dh)R_a^{(0)}(dy).\end{aligned}$$

The theorem is proved. □

Theorem 2 is of interest because $\mathbb{P}_x[\underline{X}_{\tau_a} < 0]$ is in fact the probability of ruin occurring before a critical drawdown of size a, i.e. it is the probability that the reserve falls below the level zero during the interval $[0, \tau_a]$.

In Sect. 4.1, we first give general expressions for the probability measures of depletion-related quantities. As it might be of interest, from a risk management point of view, to study the depletion problem when ruin does not occur before the critical drawdown time, we also compute the distribution of depletion-related quantities on the event $\{\tau_a < \tau\}$. This is carried out in Sect. 4.2.

4.1 Distributions of depletion quantities

Theorem 3 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ with initial surplus $x \geqslant 0$ satisfying assumptions of Sect. 2 and let $a > 0$ be a fixed critical drawdown size. Then,*

1. *the probability distribution of the drawdown observed just before critical drawdown is the following,*

$$\mathbb{P}_x(Y_{\tau_a-} \in dy) = \left(R_a^{(0)}(dy)\int_0^{\infty} v(a-y+dh)\right)\mathbb{1}_{y\in(0,a]} + \Delta^{(0)}(a)\delta_a(dy)\,, \tag{18}$$

2. *the probability distribution of the overshoot over the critical drawdown $Y_{\tau_a} - a$ is the following,*

$$\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \int_0^a \nu(a - y + dh) R_a^{(0)}(dy) \mathbb{1}_{h>0} + \Delta^{(0)}(a)\delta_0(dh) \ , \quad (19)$$

3. *the maximum reserve level attained before critical a drawdown, $\overline{X}_{\tau_a}$, follows a translated exponential distribution, i.e.*

$$\mathbb{P}_x(\overline{X}_{\tau_a} \in dv) = \lambda(a,0) e^{-\lambda(a,0)(v-x)} \mathbb{1}_{v \geqslant x} dv \ .$$

Proof In order to prove this result, we use Theorem 1 when $u \to -\infty$ with $r = q = 0$. By integrating,

1. For $y \in [0, a)$, we have
$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a -} \in dy\}}] = R_a^{(0)}(dy) \int_x^\infty F_{0,0,a}(v - x) dv \int_0^\infty \nu(a - y + dh),$$
and for $y = a$, $\mathbb{P}(Y_{\tau_a -} = a) = \Delta^{(0)}(a) \int_x^\infty F_{0,0,a}(v - x) dv$.
2. For $h > 0$, we have
$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a} - a \in dh\}}] = \int_x^\infty F_{0,0,a}(v - x) dv \int_0^a \nu(a - y + dh) R_a^{(0)}(dy),$$
and for $h = 0$, $\mathbb{P}(Y_{\tau_a} = a) = \int_x^\infty F_{0,0,a}(v - x) dv \Delta^{(0)}(a)$.
3. Finally,
$$\begin{aligned}\mathbb{P}_x(\overline{X}_{\tau_a} \in dv) &= \mathbb{E}_x\left[\mathbb{1}_{\{\overline{X}_{\tau_a} \in dv\}}\right] \\ &= F_{0,0,a}(v - x) dv \left(\int_0^a R_a^{(0)}(dy) \int_0^\infty \nu(a - y + dh) + \Delta^{(0)}(a) \right).\end{aligned}$$
Using (17) and (9) yields the result.

□

We notice from Theorem 3 that the distributions of $Y_{\tau_a^-}$ and of $Y_{\tau_a} - a$ do not depend on the initial surplus x and whatever are the characteristics of the Lévy process X, the distribution of the maximum reserve level $\overline{X}_{\tau_a}$ attained before critical drawdown is always an exponential distribution shifted by the initial surplus x. This result is a typical extension of the same result where we study the distribution of the maximum reserve level $\overline{X}_{e_q}$ attained before an exponentially distributed random time e_q with parameter q.

Now we turn our attention to the random times τ_a and $\overline{G}_{\tau_a}$. We start by giving an interesting result concerning the speed of depletion $\tau_a - \overline{G}_{\tau_a}$. This is an immediate consequence from to Theorem 1 that was not pointed out in [20], yet it is crucial in evaluating all components in the expression (13).

Proposition 1 *Under the same assumptions and definitions of Theorem 1, the random variables $\overline{G}_{\tau_a}$ and $\tau_a - \overline{G}_{\tau_a}$ are independent.*

Proof It can be easily verified that the statement in Theorem 1 still holds under weaker conditions on q and r. In fact, the result in (13) holds true for $q \geqslant 0$ and $q + r \geqslant 0$ and not only for $q, r \geqslant 0$ as indicated in the original statement. In fact, the conditions on q and r arise in the proof when we want to take the q and $q + r$ -scale functions into account in the expressions given in Theorem 1. As the scale functions $W^{(q)}, W^{(q+r)}$ are just well defined for $q, q + r \geqslant 0$.

By definition, $\overline{G}_{\tau_a}$ and $\tau_a - \overline{G}_{\tau_a}$ are positive $\mathbb{P}$-almost-surely finite random variables. It is well-known that, for $r, q \geqslant 0$, the bivariate Laplace transform $\mathbb{E}_x\left[e^{-r\overline{G}_{\tau_a}-q(\tau_a-\overline{G}_{\tau_a})}\right]$ characterizes the joint distribution of $\overline{G}_{\tau_a}$ and $\tau_a - \overline{G}_{\tau_a}$ (see for example [7]).

Clearly,

$$\mathbb{E}_x\left[e^{-r\overline{G}_{\tau_a}-q(\tau_a-\overline{G}_{\tau_a})}\right] = \mathbb{E}_x\left[e^{-q\tau_a-(r-q)\overline{G}_{\tau_a}}\right].$$

An expression for the bivariate Laplace transform of $\overline{G}_{\tau_a}$ and $\tau_a - \overline{G}_{\tau_a}$ can be obtained through identities (13) and (14) in Theorem 1. Since $F_{r,q,a}(v)$ is the product of a function depending only on r and a function depending only on q, Expressions (13) and (14) are also the product of a function depending only on r and a function depending only on q respectively, which concludes the proof. □

Proposition 2 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ with initial surplus $x \geqslant 0$ satisfying the assumptions of Sect. 2 and let $a > 0$ be a fixed critical drawdown size. Then, for $q \geqslant 0$, $q + r \geqslant 0$, the bivariate Laplace transform of τ_a and $\overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q\tau_a-r\overline{G}_{\tau_a}}\right] = \frac{\lambda(a,q)}{\lambda(a,q+r)}\left(\int_{y\in[0,a]}\int_{h>0} \nu(a-y+dh)R_a^{(q)}(dy) + \Delta^{(q)}(a)\right). \quad (20)$$

Proof Now, just like in the proof of Proposition 1, we notice that the result of Theorem 1 is still valid with $q \geq 0$ and $r + q \geq 0$. Taking $u \to -\infty$ and integrating (13) and (14) with respect to v, h and y in Theorem 1, we obtain

$$\mathbb{E}_x\left[e^{-q\tau_a-r\overline{G}_{\tau_a}}\right] = \int_{v\geqslant x} F_{q+r,q,a}(v-x)dv\left(\int_{y\in[0,a]}\int_{h>0} \nu(a-y+dh)R_a^{(q)}(dy) + \Delta^{(q)}(a)\right).$$

From the definition (9) of $F_{q+r,q,a}(.)$, we have $\int_x^\infty F_{q+r,q,a}(v-x)dv = \frac{\lambda(a,q)}{\lambda(a,q+r)}$. Substituting this last equation yields,

$$\mathbb{E}_x\left[e^{-q\tau_a-r\overline{G}_{\tau_a}}\right] = \frac{\lambda(a,q)}{\lambda(a,q+r)}\left(\int_{y\in[0,a]}\int_{h>0} \nu(a-y+dh)R_a^{(q)}(dy) + \Delta^{(q)}(a)\right).$$

□

Remark 1 Putting $r = 0$ in Eq. (20), the Laplace transform of τ_a is given by

$$\mathbb{E}_x[e^{-q\tau_a}] = \int_{y\in[0,a]} \int_{h\,>\,0} \nu(a-y+dh)R_a^{(q)}(dy) + \Delta^{(q)}(a)\ .$$

Using Eq. (20) with $q=0$ and Eq. (17) the Laplace transform of $\overline{G}_{\tau_a}$ is given by

$$\mathbb{E}_x\left[e^{-r\overline{G}_{\tau_a}}\right] = \frac{\lambda(a,0)}{\lambda(a,r)}\ . \tag{21}$$

In the following, we are going to provide an expression for the Laplace transform of the depletion random variable, $\tau_a - \overline{G}_{\tau_a}$.

Theorem 4 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ with initial surplus $x \geqslant 0$ satisfying the assumptions of Sect. 2 and let $a>0$ be a fixed critical drawdown size. Then, the Laplace transform of the speed of depletion $\tau_a - \overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q(\tau_a-\overline{G}_{\tau_a})}\right] = \frac{\lambda(a,q)}{\lambda(a,0)}\left(\int_{y\in[0,a]} \int_{h\,>\,0} \nu(a-y+dh)R_a^{(q)}(dy) + \Delta^{(q)}(a)\right)\ .$$

Proof By Proposition 1, we know that $\overline{G}_{\tau_a}$ and $\tau_a - \overline{G}_{\tau_a}$ are independent variables, then for $r, q \geqslant 0$

$$\mathbb{E}_x\left[e^{-r\overline{G}_{\tau_a}}\right]\mathbb{E}_x\left[e^{-q(\tau_a-\overline{G}_{\tau_a})}\right] = \mathbb{E}_x\left[e^{-r\overline{G}_{\tau_a}-q(\tau_a-\overline{G}_{\tau_a})}\right] = \mathbb{E}_x\left[e^{-q\tau_a-(r-q)\overline{G}_{\tau_a}}\right]\ . \tag{22}$$

We can now find an expression for the right-end of Eq. (22) by setting $q^* = q$ and $r^* = r-q$ and using Eq. (20). In other words, since $q^* \geqslant 0$ and $q^*+r^* \geqslant 0$, we can then write

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\tau_a-(r-q)\overline{G}_{\tau_a}}\right] &= \mathbb{E}_x\left[e^{-q^*\tau_a-r^*\overline{G}_{\tau_a}}\right]\\ &= \frac{\lambda(a,q^*)}{\lambda(a,q^*+r^*)}\left(\int_{y\in[0,a]} \int_{h\,>\,0} \nu(a-y+dh)R_a^{(q^*)}(dy) + \Delta^{(q^*)}(a)\right)\ .\end{aligned}$$

Substituting $q^* = q$ and $r^* = r-q$ into Eq. (24) and using Eqs. (22) and (21) in Proposition 2 yield the result. □

4.2 Distributions of depletion quantities in risk management

In this subsection, we provide expressions for the conditional distribution of depletion quantities discussed in Sect. 4.1 given the event $\left\{\underline{X}_{\tau_a} \geqslant 0\right\}$. This set guarantees that ruin does not occur before the critical drawdown time. We use the notations $\mathbb{P}(.\,;A)$ and $\mathbb{E}[.\,;A]$ for $\mathbb{P}(.\cap A)$ and $\mathbb{E}[.\,\mathbb{1}_A]$ respectively.

Proposition 3 *Consider an insurance risk process* $(X_t)_{t\geqslant 0}$ *satisfying assumptions of Sect.* 2 *with initial surplus* $x>0$ *and let* $a>0$ *be a fixed critical drawdown size. Then,*

1. *the conditional distribution of the drawdown observed just before critical drawdown given the event* $\{\underline{X}_{\tau_a}\geqslant 0\}$ *is*

$$\mathbb{P}_x\left(Y_{\tau_a}-\in dy|\underline{X}_{\tau_a}\geqslant 0\right)=\frac{R_a^{(0)}(dy)e^{\lambda(a,0)x\vee a}}{C(a,x)}$$
$$\int_{h>0}e^{-\lambda(a,0)(x\vee(h+a))}\nu(a-y+dh)\mathbb{1}_{y\in[0,a]}+\frac{\Delta^{(0)}(a)}{C(a,x)}\delta_a(dy)\ ,$$

2. *the conditional distribution of the overshoot over the critical drawdown* $Y_{\tau_a}-a$ *given the event* $\{\underline{X}_{\tau_a}\geqslant 0\}$ *is*

$$\begin{aligned}&\mathbb{P}_x\left(Y_{\tau_a}-a\in dh|\underline{X}_{\tau_a}\geqslant 0\right)\\&=\frac{e^{-\lambda(a,0)(x\vee(h+a)-x\vee a)}}{C(a,x)}\int_0^a\nu(a-y+dh)R_a^{(0)}(dy)\mathbb{1}_{h>0}+\frac{\Delta^{(0)}(a)}{C(a,x)}\delta_a(dh),\end{aligned}$$

3. *the conditional distribution of the maximum reserve level attained before critical drawdown of size a given the event* $\{\underline{X}_{\tau_a}\geqslant 0\}$ *is*

$$\mathbb{P}_x\left(\overline{X}_{\tau_a}\in dv|\underline{X}_{\tau_a}\geqslant 0\right)=\frac{\lambda(a,0)e^{-\lambda(a,0)(v-x\vee a)}}{C(a,x)}$$
$$\left(\int_0^{v-a}\int_0^a\nu(a-y+dh)R_a^{(0)}(dy)+\Delta^{(0)}(a)\right)\mathbb{1}_{v\geqslant x\vee a}dv,$$

with $C(a,x):=1-\int_{y\in[0,a]}\int_{h>0}\left(1-e^{-\lambda(a,0)h}\right)\nu(x\vee a-y+dh)R_a^{(0)}(dy)$.

Proof It is clear from Theorem 2 that

$$\begin{aligned}&\mathbb{P}_x(\underline{X}_{\tau_a}\geqslant 0)=\frac{W(x\wedge a)}{W(a)}\\&\left(1-\int_{y\in[0,a]}\int_{h>0}\left(1-e^{-\lambda(a,0)h}\right)nu(x\vee a-y+dh)R_a^{(0)}(dy)\right).\end{aligned}\tag{23}$$

Using Theorem 1 with $u=0$ and $r=q=0$ yields:

1. For $y \in [0, a)$, we have

$$\mathbb{P}_x(Y_{\tau_a} - \in dy; \underline{X}_{\tau_a} \geqslant 0) = R_a^{(0)}(dy) \frac{W(x \wedge a)}{W(a)} \int_{x \vee a}^{\infty} F_{0,0,a}(v - (x \vee a)) \int_{h \in (0, v-a]} v(a - y + dh) dv, \tag{24}$$

and for $y = a$, $\mathbb{P}\big(Y_{\tau_a -} = a; \underline{X}_{\tau_a} \geqslant 0\big) = \frac{W(x \wedge a)}{W(a)} \Delta^{(0)}(a) \int_{x \vee a}^{\infty} F_{0,0,a}(v - x \vee a) dv$. By applying Fubini's Theorem and Eq. (9) together into Eq. (24) we have

$$\begin{aligned} &\mathbb{P}_x\big(Y_{\tau_a -} \in dy; \underline{X}_{\tau_a} \geqslant 0\big) \\ &= \frac{W(x \wedge a)}{W(a)} \Big[R_a^{(0)}(dy) e^{\lambda(a,0) x \vee a} \\ &\int_{h > 0} e^{-\lambda(a,0)(x \vee (h+a))} v(a - y + dh) \mathbb{1}_{y \in [0,a]} + \Delta^{(0)}(a) \delta_a(dy)\Big]. \end{aligned} \tag{25}$$

The first part of the Theorem is obtained by using Eqs. (25) and (23).

2. For $h > 0$, we have

$$\mathbb{E}_x\big[\mathbb{1}_{\{Y_{\tau_a} - a \in dh\}}; \underline{X}_{\tau_a} \geqslant 0\big] = \frac{W(x \wedge a)}{W(a)} \int_{x \vee (h+a)}^{\infty} F_{0,0,a}(v - x \vee a) dv \int_0^a v(a - y + dh) R_a^{(0)}(dy), \tag{26}$$

and for $h = 0$, $\mathbb{P}\big(Y_{\tau_a} = a; \underline{X}_{\tau_a} \geqslant 0\big) = \frac{W(x \wedge a)}{W(a)} \int_{x \vee a}^{\infty} F_{0,0,a}(v - x \vee a) dv \Delta^{(0)}(a)$. The proof of the second part of the Theorem is done by applying the last equation, (26) and (23) into the definition of conditional distribution.

3. At the end, for $v \geqslant x \vee a$

$$\mathbb{P}_x(\overline{X}_{\tau_a} \in dv; \underline{X}_{\tau_a} \geqslant 0) = \frac{W(x \wedge a)}{W(a)} F_{0,0,a}(v - x \vee a) \left(\int_0^{v-a} \int_0^a v(a - y + dh) R_a^{(0)}(dy) + \Delta^{(0)}(a)\right) dv. \tag{27}$$

The proof is complete if we use (27) and (23).

$\square$

Proposition 4 *Consider an insurance risk process $(X_t)_{t \geqslant 0}$ satisfying assumptions of Sect. 2 with initial surplus $x > 0$ and let $a > 0$ be a fixed critical drawdown size. Then, for $q, r \geqslant 0$, we have*

$$
\begin{aligned}
&\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}; \underline{X}_{\tau_a} \geqslant 0\right] \\
&= \frac{W^{(q+r)}(x \wedge a)}{W^{(q+r)}(a)} \frac{\lambda(a,q)}{\lambda(a,q+r)} \\
&\left(\int_{y\in[0,a]} \int_{h>0} e^{-\lambda(a,q+r)(x\vee(h+a)-x\vee a)} v(a-y+dh) R_a^{(q)}(dy) + \Delta^{(q)}(a)\right). \quad (28)
\end{aligned}
$$

Proof This result can be obtained, like in the proof of the previous proposition, by putting $u = 0$ in (15) and (16) in Theorem 1 and integrating with respect to dv, dh and dy. □

We notice that in general on the event $\{\underline{X}_{\tau_a} \geqslant 0\}$, $\tau_a - \overline{G}_{\tau_a}$ and $\overline{G}_{\tau_a}$ are no more independent variables (especially when the diffusion coefficient σ is positive).

5 Examples of Lévy insurance risk processes

We study in this section particular examples of risk process X satisfying the general setting described in Sect. 2 for which the q-scale function has a tractable form.

We start by studying three examples of risk process X, without Brownian motion part, i.e. $\sigma = 0$ in its Laplace exponent (5). Thus the results in Sect. 4 apply to our problem and they endow us with tools to fully study the depletion problem. Since $\sigma = 0$ in the studied models without Brownian perturbation, the set A_c in Theorem 1 is empty and the coefficient $\Delta^{(q)}(a)$ will not appear in the sequel. We also discuss two examples of risk process X starting at an initial surplus $x \geq 0$, with a Brownian motion part, i.e. $\sigma \neq 0$ in its Laplace exponent (5).

In this paper, we aim at computing expressions for the distribution of the depletion-related random variables for relevant insurance risk processes. As it turns out, the results in Theorems 1, 2, 3 and 4 lead to explicit expressions for the distribution of depletion-related random variables when the q-scale function of the model has a tractable form. In fact, we will see that a tractable form for the q-scale function is inherited by the functions λ, $F_{p,q,a}$ and $R_a^{(q)}$ defined in (8), (9) and (10) respectively. These functions are the key ingredients in the general expressions of Theorems 1, 2, 3 and 4. In this section, we show how there are some interesting examples of insurance models with tractable q-scale functions leading to relatively simple expressions for the distributions of depletion random variables. In the following, we will analyze in more detail some models for which we can have an explicit understanding of the depletion problem:

- the Classical Cramér-Lundberg model with exponential claims,
- the Gamma risk process,
- the Spectrally Negative Stable risk process,

- the Brownian perturbed model without claims,
- the Meromorphic risk process (Beta process).

5.1 Classical Cramér–Lundberg model with exponential claims

The so-called classical or Cramér-Lundberg model was introduced in [19]. The risk process is a compound Poisson process starting at $x \geqslant 0$, i.e.,

$$X_t = x + ct - \sum_{i=1}^{N_t} Z_i, \tag{29}$$

where the number of claims is assumed to follow a Poisson process $(N_t)_{t\geqslant 0}$ with intensity λ which is independent of the positive and *iid* random variables $(Z_n)_{n\geqslant 1}$ representing claim sizes. The loaded premium c is of the form $c = (1+\theta)\lambda\mathbb{E}[Z_1]$ for some safety loading factor $\theta > 0$. The form of the q-scale function in this model is relatively simple when the claim sizes are exponentially distributed with mean $1/\mu$. In this case, the Lévy measure takes the simple form $v(dx) = \lambda\mu e^{\mu x}dx$. In turn, the Laplace exponent in (5) becomes

$$\psi(s) = c\,s + \frac{\lambda s}{\mu + s}\,, \qquad s > 0. \tag{30}$$

In this case, the premium rate is $c = \lambda(1+\theta)/\mu$ where $\theta > 0$ is a positive security loading.

A path of a such process is linear by parts, so its corresponding drawdown process is quite simple to draw.

This model has been for long a textbook example for which the distribution of ruin-related quantities can be explicitly computed. Here, we study in detail the depletion problem for this particular example. Moreover, we derive explicit expressions for the distributions of depletion random variables of Theorems 1, 2, 3 and 4. In Fig. 2, we show a simulated path of this model illustrating all depletion quantities introduced in this section.

The expression for the q-scale function in this case is known and is given by

$$W^{(q)}(x) = \frac{\mu + \Phi(q)}{\eta_q} e^{\Phi(q)x} - \frac{\mu + \Theta(q)}{\eta_q} e^{\Theta(q)x} \tag{31}$$

where $\Phi(q)$ and $\Theta(q)$ are the solutions of $\psi(s) = q$, i.e. $\Phi(q) = \frac{1}{2c}\left(q + \lambda - c\mu + \eta_q\right)$ and $\Theta(q) = \frac{1}{2c}\left(q + \lambda - c\mu - \eta_q\right)$ with $\eta_q = \sqrt{(q + \lambda - c\mu)^2 + 4q\mu c}$. For details see [15].

Now we also need expressions for the functions λ and $R_a^{(q)}$. These are given in the following result.

Proposition 5 *Consider the process $(X_t)_{t\geq 0}$ in Eq. (29) where the Z_i 's are identically independent exponential random variables with mean $1/\mu$ and let $a > 0$ be a critical drawdown size. Then*

$$\begin{aligned}\lambda(a,q) &= \Phi(q) + \frac{\lambda\mu\eta_q}{c^2(\Phi(q)+\mu)^2 e^{\eta_q a/c} - c\lambda\mu}\ , \\ R_a^{(q)}(dy) &= \frac{1}{c\lambda(a,q)}\delta_0(dy) + \left[\frac{1}{\lambda(a,q)}W'^{(q)}(y) - W^{(q)}(y)\right]\mathbb{1}_{y\in(0,a]}dy\ ,\end{aligned} \tag{32}$$

with $W^{(q)}$ and $W'^{(q)}$ given by (31) and (33) respectively.

Proof By definition

$$\lambda(a,q) = \frac{W_+'^{(q)}(a)}{W^{(q)}(a)}\ .$$

By referring to Eq. (31) we can see that $W^{(q)}$ is a differentiable function on $(0,\infty)$ so by differentiating Eq. (31) we can directly obtain

$$\begin{aligned}W'^{(q)}(x) =&\ \Phi(q)W^{(q)}(x) + (\Phi(q) - \Theta(q))\frac{\mu+\Theta(q)}{\eta_q}e^{\Theta(q)x} \\ =&\ \Phi(q)W^{(q)}(x) + \frac{\mu+\Theta(q)}{c}e^{\Theta(q)x}.\end{aligned} \tag{33}$$

Combining Eqs. (31) and (33) in the definition (8) yields the first result.

As $W^{(q)}$ is an increasing function and has a mass at $x=0$, recall that in this case (see [17]), $W^{(q)}(0^+) = 1/c$, we have $W^{(q)}(dx) = W'^{(q)}(x)dx + \frac{1}{c}\delta_0(dy)$. Direct substitution into the definition (10) of $R_a^{(q)}$ yields the second result. □

The main results regarding depletion-related quantities are given in terms of $W^{(q)}$ and $\lambda(a,q)$ with $q=0$. These expressions take on a more simple form in this case and are given in the following result.

Proposition 6 *Consider the process $(X_t)_{t\geq 0}$ in (29) where the Z_i 's are identically independent exponential random variables with mean $1/\mu$ and let $a>0$ be a critical drawdown size. Then*

$$\begin{aligned}W(x) &= \frac{\mu}{\lambda(1+\theta)\theta}\left(1+\theta-e^{\frac{-\mu\theta}{1+\theta}x}\right)\ , \\ W'(x) &= \frac{\mu^2}{\lambda(1+\theta)^2}\ e^{\frac{-\mu\theta}{1+\theta}x}\ , \\ \lambda(a,0) &= \frac{\mu\theta}{(1+\theta)\left((1+\theta)e^{\frac{\mu\theta}{1+\theta}a}-1\right)}\ , \\ F_{0,0,a}(y) &= \frac{\mu\theta}{(1+\theta)\left((1+\theta)e^{\frac{\mu\theta}{1+\theta}a}-1\right)}\exp\left(\frac{-\mu\theta y}{(1+\theta)\left((1+\theta)e^{\frac{\mu\theta}{1+\theta}a}-1\right)}\right)\ , \\ R_a^{(0)}(dy) &= \frac{\mu}{\lambda\theta}\left(e^{\frac{\mu\theta}{1+\theta}(a-y)}-1\right)\mathbb{1}_{y\in(0,a]}dy + \frac{1}{\lambda\theta}\left((1+\theta)e^{\frac{\mu\theta}{1+\theta}a}-1\right)\delta_0(dy)\ .\end{aligned}$$

Proof It is straight forward by setting $q = 0$ in Proposition 5 with $c = \frac{\lambda(1+\theta)}{\mu}$, and by using definitions (9) and (10). Simply recall that $\Phi(0) = 0$, $\Theta(0) = -\theta\mu/(1+\theta)$ and $\eta_0 = \lambda\theta$. □

We now give explicit representations for the distributions in Theorems 2, 3, and 4 as they are specialized to this case.

Proposition 7 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ of the form defined in (29) with an initial level $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \frac{W(x \wedge a)}{W(a)}\left(1 - \frac{\lambda(a,0)}{\mu + \lambda(a,0)} e^{-\mu(x \vee a - a)}\right), \tag{34}$$

where $\lambda(a,0)$ and the scale function W are given in Proposition 6.

Proof To show this proposition we use the expression of $\mathbb{P}_x\left(\underline{X}_{\tau_a} < 0\right)$ given in Theorem 2. But in this model $\sigma = 0$ so we have $\Delta^{(0)}(a) = 0$ and

$$\begin{aligned}\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \frac{W(x \wedge a)}{W(a)} + \frac{W(x \wedge a)}{W(a)} \\ \int_{y \in [0,a]} \int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh) R_a^{(0)}(dy),\end{aligned} \tag{35}$$

The Lévy measure for the process S_t is $\nu(dx) = \lambda\mu e^{-\mu x} dx$. So, by replacing this into the interior integral in (35) we have

$$\begin{aligned}\int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh) &= \int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \lambda\mu e^{-\mu(x \vee a - y + h)} dh \\ &= \lambda e^{-\mu(x \vee a)} \left(e^{\mu y} - \frac{\mu}{\mu + \lambda(a,0)} e^{\mu y}\right).\end{aligned} \tag{36}$$

On the other hand, we have

$$\lambda \int_0^a e^{\mu y} R_a^{(0)}(dy) = e^{\mu a}. \tag{37}$$

So by applying Eqs. (37) and (36) into Eq. (35) we have

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \frac{W(x \wedge a)}{W(a)} + \frac{W(x \wedge a)\lambda(a,0)e^{-\mu(x \vee a - a)}}{W(a)(\mu + \lambda(a,0))}.$$

□

Proposition 8 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ of the form defined in Eq. (29) with an initial level $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

1. *the largest drawdown observed before critical drawdown follows a mixture of a diffusive distribution on* $(0, a]$ *and the Dirac measure at* 0. *i.e.,*

$$\mathbb{P}_x(Y_{\tau_a-} \in dy) = \frac{\mu}{\theta}\left(e^{-\frac{\mu}{1+\theta}(a-y)} - e^{-\mu(a-y)}\right)\mathbb{1}_{y\in(0,a]}dy$$
$$+\frac{1}{\theta}\left((1+\theta)e^{-\frac{\mu}{1+\theta}a} - e^{-\mu a}\right)\delta_0(dy)\ ,$$

2. *the overshoot over the critical drawdown* $Y_{\tau_a} - a$ *follows an exponential distribution with mean* $1/\mu$. *i.e.,*

$$\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \mu e^{-\mu h}\mathbb{1}_{h>0}dh\ .$$

Proof We prove this proposition by applying Theorem 3. Here $\sigma = 0$ and naturally $\Delta^{(0)}(a) = 0$.

1. By plugging $\int_0^\infty v(a-y+dh) = \lambda e^{-\mu(a-y)}$ into (18) we have $\mathbb{P}_x(Y_{\tau_a-} \in dy) = \lambda e^{-\mu(a-y)}R_a^{(0)}(dy)$. We conclude the first part of the theorem using the expression of $R_a^{(0)}(dy)$ given in Proposition 6.
2. To prove the second part of the theorem we have

$$v(a-y+dh) = \lambda\mu e^{-(a-y+h)\mu}dh. \tag{38}$$

for $h>0, y\in[0,a]$. Substituting (38) in (19) gives $\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \mu e^{-\mu h}dh$.

□

We now provide explicit expressions for joint Laplace transform of $\overline{G}_{\tau_a}$ and τ_a as well as the Laplace transform of the speed of depletion. Notice that the Laplace transform of $\overline{G}_{\tau_a}$ is a simple function of $\lambda(a,q)$ and it can be computed in a straightforward way using Proposition 2 and Eq. (32).

Proposition 9 *Consider an insurance risk process* $(X_t)_{t\geqslant 0}$ *of the form defined in* (29) *with an initial level* $x \geqslant 0$ *and let* $a > 0$ *be a fixed critical drawdown size. Then, for* $q, r \geqslant 0$

1. *the bivariate Laplace transform of* τ_a *and* $\overline{G}_{\tau_a}$ *is given by*

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \left(\frac{\Phi(q)+\mu}{\mu}\right)\left(\frac{\lambda(a,q)-\Phi(q)}{\lambda(a,q+r)}\right)e^{\Phi(q)a}\ , \tag{39}$$

with $\Phi(q) = (2c)^{-1}\left(q+\lambda-c\mu+\sqrt{(q+\lambda-c\mu)^2+4q\mu c}\right)$,

2. *the Laplace transform of the speed of depletion* $\tau_a - \overline{G}_{\tau_a}$ *is given by*

$$\mathbb{E}_x\left[e^{-q(\tau_a-\overline{G}_{\tau_a})}\right] = \left(\frac{\Phi(q)+\mu}{\mu}\right)\left(\frac{\lambda(a,q)-\Phi(q)}{\lambda(a,0)}\right)e^{\Phi(q)a}\ .$$

Recall that $\lambda(a,.)$ *is given in Proposition* 6.

We observe in this model, by taking $q \to +\infty$ in the Laplace transform of $\tau_a - \overline{G}_{\tau_a}$, that with positive probability the random times are equals,

$$\mathbb{P}(\tau_a = \overline{G}_{\tau_a}) = \frac{\lambda}{c\lambda(a,0)}e^{-\mu a} = \frac{1+\theta}{\theta}\left(e^{\frac{-\mu a}{1+\theta}} - e^{-\mu a}\right).$$

Proof As $\sigma = 0$, then $\Delta^{(0)}(a) = 0$.

1. The Lévy measure of the process X is $\nu(dx) = \lambda\mu e^{-\mu x}dx$ and so $\int_0^\infty \nu(a-y+dh) = \lambda e^{-\mu(a-y)}$. Now using Proposition 2 as it specializes to this case yields

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{\lambda(a,q)\,\lambda e^{-\mu a}}{\lambda(a,q+r)}\int_{y\in[0,a]} e^{\mu y}R_a^{(q)}(dy)\ . \tag{40}$$

Now, using Proposition 5 we can compute the following integral.

$$\lambda e^{-\mu a}\int_{y\in[0,a]} e^{\mu y}R_a^{(q)}(dy) = \frac{\lambda}{\lambda(a,q)}W^{(q)}(a) - \lambda e^{-\mu a}\left(\frac{\mu}{\lambda(a,q)}+1\right)\int_0^a e^{\mu y}W^{(q)}(y)dy. \tag{41}$$

In order to finish the proof it would be sufficient to find an expression for $\int_0^a e^{\mu y}W^{(q)}(dy)$. By using Eq. (31), we have

$$\int_0^a e^{\mu y}W^{(q)}(y)dy = \frac{1}{\eta_q}\left[e^{(\mu+\Phi(q))a} - e^{(\mu+\Theta(q))a}\right]\ . \tag{42}$$

Now, by combining Eqs. (42) and (41) into (40) we obtain

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{\lambda}{\lambda(a,q+r)}\left(W^{(q)}(a) - \frac{(\mu+\lambda(a,q))}{\eta_q}\left(e^{\Phi(q)a} - e^{\Theta(q)a}\right)\right).$$

Using the expressions (31) and (32) for $W^{(q)}(a)$ and $\lambda(a,q)$ respectively, we deduce that

$$\begin{aligned}\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] &= \frac{\lambda}{\lambda(a,q+r)}\left(\frac{\Phi(q)-\lambda(a,q)}{\eta_q}e^{\Phi(q)a} + \frac{\lambda(a,q)-\Theta(q)}{\eta_q}e^{\Theta(q)a}\right)\\ &= \frac{\lambda(a,q)-\Phi(q)}{\lambda(a,q+r)}\frac{\Phi(q)+\mu}{\mu}e^{\Phi(q)a}.\end{aligned}$$

This completes the proof.

2. In a similar way, specializing Theorem 4 to this case using the expressions in Proposition 5 yields the result.

□

In the following proposition, we provide the conditional distribution for the depletion quantities given the event $\{\underline{X}_{\tau_a} \geqslant 0\}$. Expressing the joint Laplace transform for $\overline{G}_{\tau_a}$ and τ_a in the presence of the event $\{\underline{X}_{\tau_a} \geqslant 0\}$ is also of interest so that for which we discuss in the following proposition.

Proposition 10 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ of the form defined in (29) with an initial level $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

1. $\mathbb{P}_x\left(Y_{\tau_a-} \in dy | \underline{X}_{\tau_a} \geqslant 0\right) = \lambda e^{-\mu(a-y)} R_a^{(0)}(dy) \mathbb{1}_{[0,a]}(y)$,
2. $\mathbb{P}_x\left(Y_{\tau_a} - a \in dh | \underline{X}_{\tau_a} \geqslant 0\right) = \frac{\mu(\mu+\lambda(a,0))e^{-\lambda(a,0)(x\vee(h+a)-x\vee a)-\mu h}dh}{\mu+\lambda(a,0)(1-e^{-\mu(x-a)})\vee 0} \mathbb{1}_{h>0}$,
3. $\mathbb{P}_x\left(\overline{X}_{\tau_a} \in dv | \underline{X}_{\tau_a} \geqslant 0\right) = \frac{\lambda(a,0)(\mu+\lambda(a,0))e^{-\lambda(a,0)x\vee a}}{\mu+\lambda(a,0)(1-e^{-\mu(x-a)})\vee 0} e^{-\lambda(a,0)v}\left(1 - e^{-\mu(v-a)}\right)\mathbb{1}_{v\geqslant x\vee a} dv$,
4. $\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}; \underline{X}_{\tau_a} \geqslant 0\right] = \frac{W^{(q+r)}(x\wedge a)}{W^{(q+r)}(a)} \frac{(\mu+\Phi(q))(\lambda(a,q)-\Phi(q))}{\mu\lambda(a,q+r)}\left(1 - \frac{\lambda(a,q+r)}{\mu+\lambda(a,q+r)} e^{-\mu(x\vee a-a)}\right)$.

Proof To prove this proposition we need to use Proposition 3 and 4.

1. To show the first part we compute the expression given in (25). In fact by considering different cases between x and a we have

$$\mathbb{P}_x\left(Y_{\tau_a-} \in dy; \underline{X}_{\tau_a} \geqslant 0\right) = \frac{W(x\wedge a)}{W(a)} \times \left(\lambda e^{-\mu(a-y)} R_a^{(0)}(dy)\left(1 - \frac{\lambda(a,0)}{\mu+\lambda(a,0)} e^{-\mu(x\vee a-a)}\right)\right). \tag{43}$$

On the other side, by (34) we have

$$\mathbb{P}_x\left(\underline{X}_{\tau_a} \geqslant 0\right) = \frac{W(x\wedge a)}{W(a)}\left(1 - \frac{\lambda(a,0)}{\mu+\lambda(a,0)} e^{-\mu(x\vee a-a)}\right) \tag{44}$$

Applying (44) and (43) to (25) yields the result in the first part of the theorem.

2. To show this part we also compute the expression given in (26). After simplifying the expression we have

$$\mathbb{P}_x\left(Y_{\tau_a} - a \in dh; \underline{X}_{\tau_a} \geqslant 0\right) = \frac{W(x\wedge a)}{W(a)}\left(\mu e^{-\lambda(a,0)(x\vee(h+a)-x\vee a)-\mu h}\right) dh. \tag{45}$$

One again applying (45) and (44) to (26) proves the result in the second part of the theorem.

3. This part can be proven by computing the expression given in (27) and using (44). In fact after some simplifications we get

$$\mathbb{P}_x\left(\overline{X}_{\tau_a}\in dv;\underline{X}_{\tau_a}\geqslant 0\right)=\frac{W(x\wedge a)}{W(a)}$$
$$\left(\lambda(a,0)e^{-\lambda(a,0)x\vee a}\left(e^{-\lambda(a,0)v}-e^{-(\lambda(a,0)+\mu)v+\mu a}\right)\right)dv. \tag{46}$$

To end the proof we just need to replace Eqs. (46) and (44) to Eq. (27).

4. The last part of the theorem can be shown directly by computing the expression in (28). It is clear from Eq. (39) that

$$\lambda\int_{y\in[0,a]}e^{\mu y}R_a^{(q)}(dy)=\left(\frac{\Phi(q)+\mu}{\mu}\right)\left(\frac{\lambda(a,q)-\Phi(q)}{\lambda(a,q)}\right)e^{(\Phi(q)+\mu)a}. \tag{47}$$

So, the proof is complete if we apply Eq. (47) to the expression in Eq. (28) and simplify the expression.

Remark 2 Under the same assumptions of Proposition 10, it can be seen that

1. from the first part of Proposition 10, the event $\{Y_{\tau_a-}\in dy\}$ is independent of the event $\{\underline{X}_{\tau_a}\geqslant 0\}$. In fact we have $\mathbb{P}_x\left(Y_{\tau_a-}\in dy|\underline{X}_{\tau_a}\geqslant 0\right)=\lambda e^{-\mu(a-y)}R_a^{(0)}(dy)=\mathbb{P}_x(Y_{\tau_a-}\in dy)$. Thus, knowing $\underline{X}_{\tau_a}$ does not affect on the distribution of the largest drawdown before critical drawdown, Y_{τ_a-}.
2. from the second part of Proposition 10, if $x<a$, then the random variable $Y_{\tau_a}-a$ given $\{\underline{X}_{\tau_a}\geqslant 0\}$ follows an exponential distribution with parameter $\mu+\lambda(a,0)$. In other words,

$$\mathbb{P}_x\left(Y_{\tau_a}-a\in dh|\underline{X}_{\tau_a}\geqslant 0\right)=(\mu+\lambda(a,0))e^{-(\mu+\lambda(a,0))h}\mathbb{1}_{h>0}.$$

3. from the joint Laplace transform of $(\tau_a,\overline{G}_{\tau_a})$, we deduce that $\overline{G}_{\tau_a}$ and $\tau_a-\overline{G}_{\tau_a}$ are independent random variables on the event $\left\{\underline{X}_{\tau_a}\geqslant 0\right\}$. Thus, the Laplace transform of the depletion random variable , $\tau_a-\overline{G}_{\tau_a}$, given the event $\left\{\underline{X}_{\tau_a}\geqslant 0\right\}$ is

$$\mathbb{E}_x\left[e^{-q\left(\tau_a-\overline{G}_{\tau_a}\right)}|\underline{X}_{\tau_a}\geqslant 0\right]=\frac{\mu+\Phi(q)}{\mu}\frac{\lambda(a,q)-\Phi(q)}{\lambda(a,0)}.$$

In fact, with the same proof as for Proposition 1 we can show that $\overline{G}_{\tau_a}$ and $\tau_a-\overline{G}_{\tau_a}$ are independent random variables on the event $\left\{\underline{X}_{\tau_a}\geqslant 0\right\}$. Moreover, we have $\mathbb{E}_x\left[e^{-q\left(\tau_a-\overline{G}_{\tau_a}\right)}|\underline{X}_{\tau_a}\geqslant 0\right]=\frac{\mathbb{E}_x\left[e^{-q\left(\tau_a-\overline{G}_{\tau_a}\right)};\underline{X}_{\tau_a}\geqslant 0\right]}{\mathbb{P}_x(\underline{X}_{\tau_a}\geqslant 0)}$, where $\mathbb{E}_x\left[e^{-q\left(\tau_a-\overline{G}_{\tau_a}\right)};\underline{X}_{\tau_a}\geqslant 0\right]$ can be obtained from part 4 in Proposition 10 as $\overline{G}_{\tau_a}$ and $\tau_a-\overline{G}_{\tau_a}$ are independent random variables on the event $\left\{\underline{X}_{\tau_a}\geqslant 0\right\}$. Moreover, $\mathbb{P}_x(\underline{X}_{\tau_a}\geqslant 0)$ is given by Eq. (44).

5.2 Gamma risk process

The gamma risk model was introduced in [6] and is defined by

$$X_t = x + ct - S_t \ , \tag{48}$$

where the aggregate claims process $(S_t)_{t \geqslant 0}$ is assumed to follow a gamma subordinator with Lévy measure

$$v(dx) = \alpha x^{-1} e^{-\beta x} dx \ , \qquad x > 0 \ ,$$

where $\alpha, \beta > 0$. The loaded premium c is of the form $c = (1+\theta)\mathbb{E}[S_1]$ for some safety loading factor $\theta > 0$. In turn, the Laplace exponent in (5) becomes

$$\psi_X(s) = c\, s - \alpha \ln(1 + \frac{s}{\beta}) \ . \qquad s > 0 \ ,$$

We refer the reader to [9] for a discussion of subordinator models in risk theory. In this section we are going to provide expressions for $\underline{X}_{\tau_a}, \overline{X}_{\tau_a}, Y_{\tau_a -}$ and $Y_{\tau_a} - a$ associated to the process X given by $X_t = ct - S_t$.

To find these expressions we need first to provide $W(x)$ for X. Let the process X start at $x \geqslant 0$. Based on the result given in Chapter 8 of [17] for survival probability for a spectrally negative Lévy process we have

$$1 - \phi(x) = \begin{cases} \psi_X'(0^+) W(x) & \text{if } \psi_X'(0^+) > 0 \\ 0 & \text{Otherwise,} \end{cases} \tag{49}$$

where $\phi(x)$ is the probability of ruin and ψ_X is the Laplace exponent for X. On the other hand, [6] gives another expression for survival probability for $X_t = ct - S_t$ when S is a gamma subordinator. That is

$$1 - \phi(x) = \frac{\theta}{1+\theta} \sum_{n \geqslant 0} \frac{1}{(1+\theta)^n} M^{*n}(x), \tag{50}$$

where $M(x) = \beta \int_0^x \int_{\beta t}^{\infty} u^{-1} e^{-u} du dt = 1 + \beta x E_1(\beta x)$. Here $E_1(x) = \int_x^{\infty} u^{-1} e^{-u} du$ is the exponential integral function and $M^{*n}(x) = \int_0^x M^{*(n-1)}(x-y) M'(y) dy$ is the nth-fold convolution where $M'(y) = \beta E_1(\beta y)$.

As $\psi_X'(0^+) = c - \psi_S'(0^+) = c - \frac{\alpha}{\beta}$ and $c = \frac{\alpha(1+\theta)}{\beta}$, we have $\psi_X'(0^+) = \frac{\alpha\theta}{\beta} > 0$. Now, by equalizing (49) and (50) we can get the expression for $W(x)$. More precisely, we have

$$W(x) = \frac{\beta}{(1+\theta)\alpha} \sum_{n \geqslant 0} \frac{1}{(1+\theta)^n} M^{*n}(x). \tag{51}$$

Taking derivative of $W(x)$ in (51) yields

$$\begin{aligned} W'(x) &= \frac{\beta}{(1+\theta)\alpha}\sum_{n\geqslant 0}\frac{1}{(1+\theta)^n}(M^{*n})'(x) \\ &= \frac{\beta}{(1+\theta)\alpha}\sum_{n\geqslant 0}\frac{1}{(1+\theta)^n}\int_0^x \frac{\partial M^{*(n-1)}}{\partial x}(x-y) * \ M'(y)dy. \end{aligned} \tag{52}$$

This yields to the following expression for $\lambda(a,0)$.

$$\lambda(a,0) = \frac{W'(a)}{W(a)} = \frac{\sum_{n\geqslant 0}\frac{1}{(1+\theta)^n}(M^{*n})'(a)}{\sum_{n\geqslant 0}\frac{1}{(1+\theta)^n}M^{*n}(a)}. \tag{53}$$

Using (51), (52) and (53) can also provide expressions for $F_{0,0,a}(x)$ and $R_a^{(0)}(dy)$.

Proposition 11 *Consider an insurance risk process $(X_t)_{t\geqslant 0}$ of the form defined in (48) with an initial level $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

$$\mathbb{P}_x[\underline{X}_{\tau_a} < 0] = 1 - \frac{W(x\wedge a)}{W(a)} + \frac{W(x\wedge a)e^{-\beta(x\vee a)}}{W(a)}\big(G_{\beta,\beta}(x\vee a) - G_{\beta+\lambda(a,0),\beta}(x\vee a)\big),$$

where $G_{\gamma,\beta}(v)$, $\lambda(a,0)$ and $W(a)$ are given by (54), (53) and (51) respectively.

Proof Once again like the procedure we have done in the previous subsection, to show this proposition we need to use Theorem 2. Here as $\sigma = 0$ so naturally $\Delta^{(0)}(a) = 0$. We just need to compute the integrals in the expression given for $\mathbb{P}_x[\underline{X}_{\tau_a} < 0]$ in Theorem 2. The Lévy measure for the process S_t is $v(dx) = \alpha x^{-1}e^{-\beta x}dx$. So, by replacing this into $\int_{h>0}\big(1 - e^{-\lambda(a,0)h}\big)v(x\vee a - y + dh)$ we have

$$\begin{aligned} &\int_{h>0}\Big(1 - e^{-\lambda(a,0)h}\Big)v(x\vee a - y + dh) \\ &= \alpha\int_0^\infty \Big(1 - e^{-\lambda(a,0)h}\Big)(x\vee a - y + h)^{-1}e^{-\beta(x\vee a - y + h)}dh \\ &= \alpha\Big(E_1(\beta(x\vee a - y)) - e^{\lambda(a,0)(x\vee a - y)}E_1((\beta + \lambda(a,0))(x\vee a - y))\Big), \end{aligned}$$

where E_1 is the exponential integral function.

Now, define the following function.

$$G_{\gamma,\beta}(v) = \int_0^\infty e^{-\gamma h}\int_0^a (v + h - y)^{-1}e^{\beta y}R_a^{(0)}(dy)dh, \tag{54}$$

for $\gamma, v > 0$. It is clear that $G_{\beta,\beta}(a) = \frac{e^{\beta a}}{\alpha}$ because of

$$\alpha\int_0^a E_1(\beta(a-y))R_a^{(0)}(dy) = 1. \tag{55}$$

We use Eq. (17) to get Eq. (55). So by applying numerical methods we can compute the function $G_{\gamma,\beta}$.

To end the proof it is sufficient to apply (54) in (15). Thus we have

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \frac{W(x \wedge a)}{W(a)} + \frac{W(x \wedge a)e^{-\beta x \vee a}}{W(a)} \big(G_{\beta,\beta}(x \vee a) - G_{\beta+\lambda(a,0),\beta}(x \vee a)\big).$$

□

Now, we are going to provide representations for distributions of each of the random variables $Y_{\tau_a -}$ and $Y_{\tau_a} - a$.

Proposition 12 *Consider an insurance risk process $(X_t)_{t \geqslant 0}$ of the form defined in (48) with an initial level $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

1. *the distribution of $Y_{\tau_a -}$, the largest drawdown observed before the critical drawdown of size a, is*

$$\mathbb{P}_x(Y_{\tau_a -} \in dy) = \alpha E_1(\beta(a-y))R_a^{(0)}(dy),$$

2. *the overshoot of critical drawdown over level a* is

$$\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \alpha e^{-\beta(a+h)} \int_0^a (v + h - y)^{-1} e^{\beta y} R_a^{(0)}(dy) dh.$$

Proof As $\sigma = 0$ thus $\Delta^{(0)}(a) = 0$.

1. It is clear that $\int_0^\infty v(a - y + dh) = \alpha E_1(\beta(a-y))$. By plugging it into (18) we have

$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a -} \in dy\}}] = \alpha E_1(\beta(a-y))R_a^{(0)}(dy).$$

2. we have

$$v(a - y + dh) = \alpha e^{-\beta(a-y+h)}(a - y + h)^{-1} dh. \tag{56}$$

for $h > 0$, $y \in [0, a]$. Substituting (56) in (19) gives

$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a} - a \in dh\}}] = \alpha e^{-\beta(a+h)} \int_0^a (a + h - y)^{-1} e^{\beta y} R_a^{(0)}(dy) dh.$$

□

5.3 Spectrally negative stable process

In this subsection, we study the case when X_t is a spectrally negative stable process with stability parameter $\alpha \in (1, 2)$. This model was study in the insurance context in [8]. We study the running minimum at the first-passage time over a level a of the drawdown process Y. More precisely, we give representations for distributions of each of the stochastic processes $\overline{X}_{\tau_a}, Y_{\tau_a-}$ and $Y_{\tau_a} - a$ associated to the main process X given above.

Let $(X_t)_{t\geq 0}$ be a spectrally negative stable process with stability parameter $\alpha \in (1,2)$ and Laplace exponent $\psi(s) = s^\alpha$ for $s > 0$. Moreover, the Lévy measure in (5) is $v(dx) = \frac{\beta}{x^{1+\alpha}}dx$ for $x, \beta > 0$. It can be seen (see for example [18, 20]) that

$$W^{(q)}(x) = \alpha x^{\alpha-1}E'_{\alpha,1}(qx^\alpha), \tag{57}$$

for $x, q \geqslant 0$ where $E_{\alpha,1}(z) = \sum_{k\geqslant 0}\frac{z^k}{\Gamma(1+\alpha k)}$ is the Mittag-Leffler function and Γ is the gamma function.

Proposition 13 *Let $(X_t)_{t\geq 0}$ be a stable process with stability parameter $\alpha \in (1,2)$ and let $a > 0$ be a critical drawdown size. Then*

$$\lambda(a,q) = \frac{\alpha-1}{a} + q\alpha a^{\alpha-1}\frac{E''_{\alpha,1}(qa^\alpha)}{E'_{\alpha,1}(qa^\alpha)}\ , \tag{58}$$

$$R_a^{(q)}(dy) = \left[\alpha y^{\alpha-2}E'_{\alpha,1}(qy^\alpha)\left(\frac{\alpha-1}{\lambda(a,q)} - y\right) + \frac{q\alpha^2y^{2\alpha-2}}{\lambda(a,q)}E''_{\alpha,1}(qy^\alpha)\right]dy\ . \tag{59}$$

Proof Taking derivative of (57) with respect to x and substitute it in $\lambda(a,q)$ given by (8) (in $R_a^{(q)}$ given by (10) respectively) yields (58)((59) respectively). □

As a particular case of Proposition 13, we have

$$\lambda(a,0) = \frac{\alpha-1}{a} \quad \text{and} \quad \mathrm{R}_{\mathrm{a}}^{(0)}(\mathrm{dy}) = \frac{\alpha \mathrm{y}^{\alpha-2}}{\Gamma(1+\alpha)}(\mathrm{a}-\mathrm{y})\mathrm{dy}. \tag{60}$$

Proposition 14 *Consider a spectrally negative stable process $(X_t)_{t\geqslant 0}$ with stability parameter $\alpha \in (1,2)$ with an initial surplus $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = 1 - \left(\frac{x\wedge a}{a}\right)^{\alpha-1} + \beta\left(\frac{x\wedge a}{a}\right)^{\alpha-1}\left(\frac{H_{0,0,0}(x\vee a)}{\alpha} - H_{\lambda(a,0),\lambda(a,0),0}(x\vee a)\right),$$

where $\lambda(a,0)$ is given by (58) and $g_{a,\alpha,0}(v)$ is defined by (61).

Proof To show this proposition we need to use Theorem 2 one more time. The Lévy measure for the process $-X_t$ is $v(dh) = \frac{\beta}{h^{1+\alpha}}dh$ for $\beta > 0$. So, by replacing this into $\int_{h>0}\left(1 - e^{-\lambda(a,0)h}\right)v(x\vee a - y + dh)$ we have

$$\begin{aligned}
&\int_{h>0}\left(1-e^{-\lambda(a,0)h}\right)\nu(x\vee a-y+dh)\\
&=\int_{h>x\vee a-y}\left(1-e^{\lambda(a,0)(x\vee a-y)}e^{-\lambda(a,0)h}\right)\frac{\beta}{h^{1+\alpha}}dh\\
&=\frac{\beta}{\alpha}\left[\frac{1}{(x\vee a-y)^{\alpha}}-e^{\lambda(a,0)(x\vee a-y)}\int_{h>x\vee a-y}\frac{\alpha e^{-\lambda(a,0)h}}{h^{1+\alpha}}dh\right].
\end{aligned}$$

Now define

$$H_{\gamma_1,\gamma_2,q}(v)=\int_0^{\infty}e^{-\gamma_1 h}\int_0^a\frac{e^{-\gamma_2 y}R_a^{(q)}(dy)}{(v-y+h)^{\alpha+1}}dh. \tag{61}$$

We can use the numerical methods to compute $H_{\gamma_1,\gamma_2,q}(v)$ for a given value v. For the special case $\gamma_1=\gamma_2=0$, $v=a$ and $q=0$ we have

$$H_{0,0,0}(a)=\int_0^a\frac{R_a^{(0)}(dy)}{(a-y)^{\alpha}}=\frac{\alpha}{\beta}, \tag{62}$$

because of

$$\int_0^a\int_0^{\infty}\frac{\beta}{(a-y+h)^{1+\alpha}}dhR_a^{(0)}(dy)=1. \tag{63}$$

We use (17) to get Eq. (63). To end the proof it is sufficient to replace (61) and (62) into (15).

Therefore, after some simplifications we get

$$\mathbb{P}_x(\underline{X}_{\tau_a}<0)=1-\left(\frac{x\wedge a}{a}\right)^{\alpha-1}+\beta\left(\frac{x\wedge a}{a}\right)^{\alpha-1}\left(\frac{H_{0,0,0}(x\vee a)}{\alpha}-H_{\lambda(a,0),\lambda(a,0),0}(x\vee a)\right).$$

□

Proposition 15 *Consider a spectrally negative stable process $(X_t)_{t\geqslant 0}$ with stability parameter $\alpha\in(1,2)$ with an initial surplus $x\geqslant 0$ and let $a>0$ be a fixed critical drawdown size. Then*

1. *the distribution of Y_{τ_a-}, the largest drawdown observed before the critical drawdown of size a, is*

$$\mathbb{P}_x(Y_{\tau_a-}\in dy)=\frac{\beta y^{\alpha-2}}{\Gamma(1+\alpha)(a-y)^{\alpha-1}}dy,$$

2. *the overshoot of critical drawdown over level a is*

$$\mathbb{P}_x(Y_{\tau_a}-a\in dh)=\beta\int_0^a\frac{R_a^{(0)}(dy)}{(a-y+h)^{\alpha+1}}dh.$$

Proof As $\sigma = 0$, thus $\Delta^{(0)}(a) = 0$.

1. It can be easily shown that $\int_0^\infty \nu(a - y + dh) = \frac{\beta}{\alpha(a-y)^\alpha}$. By plugging it into (18) we get

$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a-}\in dy\}}] = \frac{\beta}{\alpha(a-y)^\alpha} R_a^{(0)}(dy). \tag{64}$$

The proof of first part is done by replacing (60) into (64).

2. To prove the second part of the theorem we have

$$\nu(a - y + dh) = \frac{\beta}{(a - y + h)^{\alpha+1}} dh. \tag{65}$$

Substituting (65) in (19) gives

$$\mathbb{E}_x[\mathbb{1}_{\{Y_{\tau_a}-a\in dh\}}] = \beta \int_0^a \frac{R_a^{(0)}(dy)}{(a - y + h)^{\alpha+1}} dh.$$

□

In the sequel of this subsection, we are going to provide the joint Laplace transform of τ_a and $\overline{G}_{\tau_a}$. In fact, we are seeking the Laplace transform of the depletion random variable, $\tau_a - \overline{G}_{\tau_a}$.

Proposition 16 *Consider a spectrally negative stable process $(X_t)_{t\geqslant 0}$ with stability parameter $\alpha \in (1, 2)$ with an initial surplus $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size.*

1. *The bivariate Laplace transform of τ_a and $\overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{\beta\lambda(a,q)}{\alpha\lambda(a, q + r)} H_{0,0,q}(a),$$

where $H_{0,0,q}(a)$ is defined by (61) and $\lambda(a, q)$ is given by (58).

2. *The Laplace transform of the speed of depletion $\tau_a - \overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q(\tau_a - \overline{G}_{\tau_a})}\right] = \frac{\beta\lambda(a,q)}{\alpha\lambda(a,0)} H_{0,0,q}(a) .$$

Proof As $\sigma = 0$, thus $\Delta^{(0)}(a) = 0$ in Proposition 2 and Theorem 4. 1. The Lévy measure of the process $-X$ is $\frac{\beta}{x^{\alpha+1}} dx$ and so $\int_0^\infty \nu(a - y + dh) = \frac{\beta}{\alpha(a-y)^\alpha}$. Now using Proposition 2 as it specializes to this case yields

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{\beta\lambda(a,q)}{\alpha\lambda(a,q+r)}\int_0^a \frac{R_a^{(q)}(dy)}{(a-y)^\alpha} = \frac{\beta\lambda(a,q)}{\alpha\lambda(a,q+r)} H_{0,0,q}(a).$$

This ends the proof of the first part. 2. In a similar way, specializing Theorem 4 to this case using the expressions in (58) and (59) yields the result. □

5.4 Brownian motion with drift

The Brownian motion with drift was studied in the insurance context as an approximation to the classical Cramér-Lundberg model [11]. In our context, the insurance risk process X is a Brownian motion with drift, starting at $x \geqslant 0$, i.e.

$$X_t = x + ct + \sigma B_t\ , \tag{66}$$

where $(B_t)_{t\geqslant 0}$ is assumed to be a standard Brownian motion. The form of the q-scale function in this model is relatively simple since the claim sizes disappear. In this case, the Laplace exponent in (5) takes a simple form and becomes

$$\psi(s) = c\,s + \frac{\sigma^2}{2}s^2\ , \qquad s > 0\ . \tag{67}$$

We obtain an explicit expression for the q-scale function (see [15] for more details) as

$$\begin{aligned} W^{(q)}(x) &= \frac{1}{\sqrt{c^2+2q\sigma^2}}\left[e^{(\sqrt{c^2+2q\sigma^2}-c)\frac{x}{\sigma^2}} - e^{-(\sqrt{c^2+2q\sigma^2}+c)\frac{x}{\sigma^2}}\right] \\ &= \frac{2}{\sqrt{c^2+2q\sigma^2}} e^{-\frac{c}{\sigma^2}x}\sinh\left(\frac{\sqrt{c^2+2q\sigma^2}x}{\sigma^2}\right). \end{aligned} \tag{68}$$

Following the same order of ideas as in previous examples, we obtain explicit expressions for the functions λ, $R_a^{(q)}$ and $\Delta_a^{(q)}$. These are given in the following result.

Proposition 17 *Consider the process* $(X_t)_{t\geq 0}$ *in* (66) and let $a > 0$ *be a critical drawdown size. Then*

$$\begin{aligned} \lambda(a,q) &= -\frac{c}{\sigma^2} + \frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}\coth\left(\frac{a}{\sigma^2}\sqrt{c^2+2q\sigma^2}\right), \\ R_a^{(q)}(dy) &= \frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2\lambda(a,q)}W^{(q)}(y)\left[\coth\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}y\right) - \coth\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}a\right)\right]\mathbb{1}_{y\in(0,a]}dy\ , \\ \Delta^{(q)}(a) &= -\left(\frac{c}{2} + \frac{c^2+q\sigma^2}{\sigma^2\,\lambda(a,q)}\right)W^{(q)}(a) + \left(1+\frac{2c}{\sigma^2\,\lambda(a,q)}\right)e^{-\frac{c}{\sigma^2}a}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}a\right), \end{aligned} \tag{69}$$

with $W^{(q)}$ *is given by* (68).

Proof We have

$$\lambda(a,q)=\frac{W_+^{\prime(q)}(a)}{W^{(q)}(a)}\ .$$

By referring to Eq. (68) we can see that $W^{(q)}$ is a differentiable function on $(0,\infty)$ so by differentiating Eq. (68) we can directly obtain

$$W^{\prime(q)}(x)=-\frac{c}{\sigma^2}W^{(q)}(x)+\frac{2}{\sigma^2}e^{-\frac{c}{\sigma^2}x}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}x\right). \tag{70}$$

Applying (68) and (70) in Eq. (8) yields the first result.

As $W^{(q)}$ is an increasing function and does not have a mass at $x=0$, we have $W^{(q)}(dx)=W^{\prime(q)}(x)dx$ (see [17]). Direct substitution into the definition (10) of $R_a^{(q)}$ yields

$$\begin{aligned}R_a^{(q)}(dy)&=-\left(\frac{c}{\sigma^2\lambda(a,q)}+1\right)W^{(q)}(y)dy+\frac{2}{\lambda(a,q)\sigma^2}e^{-\frac{c}{\sigma^2}y}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}y}{\sigma^2}\right)dy\\&=\frac{2}{\sigma^2\lambda(a,q)}e^{-\frac{c}{\sigma^2}y}\sinh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}y\right)\\&\quad\times\left(\coth\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}y\right)-\coth\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}a\right)\right)dy.\end{aligned}$$

Differentiating Eq. (70) we obtain

$$\begin{aligned}W^{\prime\prime(q)}(x)=&\frac{c^2}{\sigma^4}W^{(q)}(x)-\frac{4c}{\sigma^4}e^{-\frac{c}{\sigma^2}x}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}x\right)\\&+\frac{2\sqrt{c^2+2q\sigma^2}}{\sigma^4}e^{-\frac{c}{\sigma^2}x}\sinh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}x\right)\\=&\frac{2c^2+2q\sigma^2}{\sigma^4}W^{(q)}(x)-\frac{4c}{\sigma^4}e^{-\frac{c}{\sigma^2}x}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}x\right)\ ,\end{aligned}\tag{71}$$

where in the last equality we have substituted $W^{\prime(q)}(x)$ by its expression provided in (70). Combining (70) and (71) in Eq. (11) for $\Delta^{(q)}(a)$ yields the third result. □

We notice that, the process being continuous, the largest drawdown observed before critical drawdown follows a Dirac measure at a and $Y_{\tau_a}=a$ a.s. In the following proposition, we give explicit value to have ruin before the critical drawdown for the risk model (66).

Proposition 18 *Consider a process $(X_t)_{t\geqslant 0}$ of the form defined in (66) with an initial level $x\geqslant 0$ and let $a>0$ be a fixed critical drawdown size. Then, for $x\geq a$, $\mathbb{P}_x(\underline{X}_{\tau_a}<0)=0$ and for $x<a$*

$$\mathbb{P}_x(\underline{X}_{\tau_a} < 0) = \frac{e^{-\frac{2c}{\sigma^2}x} - e^{-\frac{2c}{\sigma^2}a}}{1 - e^{-\frac{2c}{\sigma^2}a}}. \tag{72}$$

Proof The expression given for $\mathbb{P}_x\left(\underline{X}_{\tau_a} < 0\right)$ in Theorem 2 will be reduced to

$$\mathbb{P}_x\left(\underline{X}_{\tau_a} < 0\right) = 1 - \frac{W(x \wedge a)}{W(a)}, \tag{73}$$

since the Lévy measure vanishes. Moreover, $W(x)$ is given by

$$W(x) = \frac{1}{c}\left(1 - e^{-\frac{2cx}{\sigma^2}}\right). \tag{74}$$

Thus, (72) holds. □

In the following, we provide explicit expressions for joint Laplace transform of $\overline{G}_{\tau_a}$ and τ_a as well as the Laplace transform of the speed of depletion. Notice that the corresponding integral with respect to $R_a^{(q)}(\cdot)$ in Proposition 2 and Theorem 4 disappears since X_t has no jumps. In this case, these Laplace transforms are a simple function of $W^{(q)}(a)$ given in (68), and they can be computed in a straightforward way using Proposition 2 and Theorem 4.

Proposition 19 *Consider a process $(X_t)_{t \geqslant 0}$ of the form defined in* (66) with an initial level $x \geqslant 0$ *and let $a > 0$ be a fixed critical drawdown size. We have the following results for $q, r \geqslant 0$.*

1. *The bivariate Laplace transform of τ_a and $\overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{c^2 + 2q\sigma^2}{2\sigma^2}\left(\coth^2\left(\frac{\sqrt{c^2 + 2q\sigma^2}}{\sigma^2}a\right) - 1\right)\frac{W^{(q)}(a)}{\lambda(a, q + r)}, \tag{75}$$

   with $\lambda(a, q)$ given in (69).

2. *The Laplace transform of the speed of depletion $\tau_a - \overline{G}_{\tau_a}$ is given by*

$$\mathbb{E}_x\left[e^{-q(\tau_a - \overline{G}_{\tau_a})}\right] = \frac{c^2 + 2q\sigma^2}{2c}\frac{\coth^2\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}a\right) - 1}{\coth\left(\frac{c}{\sigma^2}a\right) - 1}W^{(q)}(a). \tag{76}$$

Proof 1. As the risk process X has no jumps, then Proposition 2 will be reduced to

$$\mathbb{E}_x\left[e^{-q\tau_a - r\overline{G}_{\tau_a}}\right] = \frac{\lambda(a, q)}{\lambda(a, q + r)}\Delta^{(q)}(a). \tag{77}$$

We can then compute Eq. (77) using the given representation for $\Delta^{(q)}(a)$ in Proposition 17. That is,

$$\begin{aligned}&\lambda(a,q)\Delta^{(q)}(a)\\&=-\Big(\frac{\lambda(a,q)c}{2}+\frac{c^2+q\sigma^2}{\sigma^2}\Big)W^{(q)}(a)+\Big(\lambda(a,q)+\frac{2c}{\sigma^2}\Big)e^{-\frac{c}{\sigma^2}a}\cosh\left(\frac{\sqrt{c^2+2q\sigma^2}}{\sigma^2}a\right).\end{aligned}\tag{78}$$

where in the last equality we have substituted $\lambda(a,q)$ by its expression given by (69). Thus, (75) holds.

2. In a similar way, specializing Theorem 4 to this case yield the result. □

5.5 Meromorphic *beta*-risk process

The *beta*-insurance risk model was introduced in the insurance context in [16]. It is one example of the class of meromorphic Lévy processes whose appeal lies in the semi-explicit form of their q-scale function. Here, we study a risk processes with a net aggregate claim process described by *beta*-process. It is of the form

$$X_t = x + ct - Z_t, \tag{79}$$

where the aggregate claims process $(Z_t)_{t\geqslant 0}$ has the following Lévy measure

$$\nu(dx) = \xi\,\beta\frac{e^{-(1+\alpha)\beta x}}{(1-e^{-\beta x})^{\lambda}}\,dx, \quad x>0, \tag{80}$$

with $\alpha, \beta, \xi > 0$ and $\lambda \in (1,2)\cup(2,3)$.

Parameters α and β are responsible for the rate of decay of the tail of the Lévy measure and for the shape of this measure, parameter ξ controls the overall "intensity" of jumps, while the parameter λ describes the singularity of the Lévy measure at zero and therefore controls the intensity of the small jumps. Indeed, if $\lambda \in (1,2)$, then the process has jump part of infinite activity and finite variation while if $\lambda \in (2,3)$, the jump part of the process will be of infinite variation. For a thorough discussion of the beta family of processes we refer to [13].

The loaded premium c is again of the form $c=(1+\theta)\mathbb{E}[Z_1]$ for some safety loading factor $\theta>0$. In turn, the Laplace exponent in (5) becomes,

$$\psi(s) = c\,s + \frac{1}{2}\sigma^2 s^2 + \xi\,B(1+\alpha+s/\beta, 1-\lambda) - \xi\,B(1+\alpha, 1-\lambda)\,. \qquad s>0\,, \tag{81}$$

where $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$ is the Beta function.

Notice that this family has a Gaussian component that can be switched on and off using the σ parameter in (81) making our risk model a member of the so-called perturbed family of models.

The beta process is in fact an example of a member of a larger family of processes having either a rational or meromorphic Laplace exponent which is at the heart of their tractability. As was shown in [14], many results of fluctuation theory can be given rather explicitly for this family of processes. These processes are defined by requiring that their Lévy density $\Pi(dx) = \pi(x)\,dx$ is essentially a "mixture" of exponential distributions, and in the spectrally negative case this translates into the following definition

$$\pi(x) = \mathbb{1}_{\{x>0\}} \sum_{m=1}^{\infty} b_m e^{-\rho_m x}, \tag{82}$$

where all the coefficients b_m and ρ_m are positive and $\rho_1 < \rho_2 < \ldots$. The Laplace exponent $\psi(s)$ for a process with the Lévy measure (82) is a meromorphic ($M = +\infty$) function (see [14]). In the case of the beta process, it has been shown (see [13]) that its Lévy density (80) is of the form in (82) with coefficients

$$b_m = \xi\beta \binom{m+\lambda-2}{m-1}, \quad \rho_m = \beta(\alpha+m)\,. \tag{83}$$

In the following, we give the form of the q-scale function for beta process and study depletion quantities for this process. The following lemma is taken from [16]

Lemma 1 *Consider the process $(X_t)_{t\geq 0}$ in (79). Then, for $q \geq 0$*

$$W^{(q)}(x) = \frac{e^{\Phi(q)x}}{\psi'(\Phi(q))} + \sum_{n\geq 1} \frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)}, \; x > 0, \tag{84}$$

where $\{\Phi(q), -\zeta_1, -\zeta_2, \ldots\}$ is the set of simple poles of the function $\frac{e^{zx}}{\psi(z)-q}$.

As we have $\psi(s) = \mathbb{E}(X_1)s + O(s^2)$ when $s \to 0$, it gives $\Phi(0) = 0$. By setting $K(x) = \sum_{n\geq 1} \frac{e^{-\zeta_n x}}{\psi'(-\zeta_n)}$, we have $W^{(0)}(x) = \frac{1}{\mathbb{E}(X_1)} + K(x)$. Taking derivative of $W^{(0)}(x)$ provides us $\lambda(a,0)$. That is,

$$\lambda(a,0) = \frac{W'^{(0)}(x)}{W^{(0)}(x)} = \frac{\mathbb{E}(X_1)K'(x)}{1+\mathbb{E}(X_1)K(x)}. \tag{85}$$

Using (85) and $W''^{(0)}(x) = K''(x) = \sum_{n\geq 1} \frac{\zeta_n^2 e^{-\zeta_n x}}{\psi'(-\zeta_n)}$, we can also find expressions for $F_{0,0,a}(x), R_a^{(0)}(dy)$ and $\Delta^{(0)}(a)$.

Proposition 20 *Consider a meromorphic beta spectrally negative process $(X_t)_{t\geqslant 0}$ with an initial surplus $x \geqslant 0$ and let $a > 0$ be a fixed critical drawdown size. Then*

1.

$$\begin{aligned} \mathbb{P}_x(\underline{X}_{\tau_a} < 0) &= 1 - \frac{1+\mathbb{E}(X_1)K(x\wedge a)}{1+\mathbb{E}(X_1)K(a)} \\ &+ \frac{1+\mathbb{E}(X_1)K(x\wedge a)}{1+\mathbb{E}(X_1)K(a)}[G(b,\rho,0) - G(b,\rho,\lambda(a,0))], \end{aligned} \tag{86}$$

2. *the distribution of $Y_{\tau_a -}$, the largest drawdown observed before the critical drawdown of size a, is:*

$$\mathbb{P}_x(Y_{\tau_a -} \in dy) = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m} e^{-\rho_m(a-y)} R_a^{(0)}(dy) + \Delta^{(0)}(a)\delta_a(dy),$$

3. *the overshoot of critical drawdown over level a is*:

$$\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \sum_{m=1}^{\infty} b_m e^{-\rho_m(a+h)} dh \int_0^a e^{\rho_m y} R_a^{(0)}(dy) + \Delta^{(0)}(a)\delta_0(dy),$$

where $G(c, d, k)$ is defined in (88).

Proof To show the first part of this proposition we need to use Theorem 2. We just need to compute the integrals in the expression given for $\mathbb{P}_x[\underline{X}_{\tau_a} < 0]$ in Theorem 2. The Lévy measure for the process X_t is given by (82). So, by replacing this into $\int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh)$ we have

$$\begin{aligned} &\int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh) \\ &= \int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \sum_{m=1}^{\infty} b_m e^{-\rho_m(x \vee a - y + h)} dh \\ &= \sum_{m=1}^{\infty} \left[\frac{b_m}{\rho_m} - \frac{b_m}{\lambda(a,0) + \rho_m}\right] e^{-\rho_m(x \vee a - y)}. \end{aligned} \tag{87}$$

For two sequences $c = (c_m)_{m \geq 1}$ and $d = (d_m)_{m \geq 1}$ and constant $k \in \mathbb{R}$ define

$$G(c, d, k) = \sum_{m=1}^{\infty} \frac{c_m}{d_m + k} \int_0^a e^{-\rho_m(x \vee a - y)} R_a^{(0)}(dy). \tag{88}$$

We can compute (88) numerically. Therefore, using (87) and (88) we have

$$\int_0^a \int_{h>0} \left(1 - e^{-\lambda(a,0)h}\right) \nu(x \vee a - y + dh) R_a^{(0)}(dy) = G(b, \rho, 0) - G(b, \rho, \lambda(a, 0)). \tag{89}$$

To finish the proof we need to substitute (89) in Eq. (15) in Theorem 2. That is,

$$\mathbb{P}_x[\underline{X}_{\tau_a} < 0] = 1 - \frac{1 + \mathbb{E}(X_1)K(x \wedge a)}{1 + \mathbb{E}(X_1)K(a)} + \frac{1 + \mathbb{E}(X_1)K(x \wedge a)}{1 + \mathbb{E}(X_1)K(a)} [G(b, \rho, 0) - G(b, \rho, \lambda(a, 0))].$$

To prove the second part of the proposition we use Eq. (18) in Theorem 3. We have

$$\int_{h>0} \nu(a - y + dh) R_a^{(0)}(dy) = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m} e^{-\rho_m(a-y)} R_a^{(0)}(dy). \tag{90}$$

By substituting (90) into (18) we get

$$\mathbb{P}_x(Y_{\tau_a-} \in dy) = \sum_{m=1}^{\infty} \frac{b_m}{\rho_m} e^{-\rho_m(a-y)} R_a^{(0)}(dy) + \Delta^{(0)}(a)\delta_a(dy).$$

To show the third part of the proposition we follow the same procedure as applied for the first and second parts. In fact, we use Eq. (19) in Theorem 3. In the other words, we have

$$\int_{y\in[0,a]} v(a-y+dh)R_a^{(0)}(dy) = \sum_{m=1}^{\infty} b_m e^{-\rho_m(a+h)} dh \int_0^a e^{\rho_m y} R_a^{(0)}(dy). \tag{91}$$

By substituting (91) into (19) we get

$$\mathbb{P}_x(Y_{\tau_a} - a \in dh) = \sum_{m=1}^{\infty} b_m e^{-\rho_m(a+h)} dh \int_0^a e^{\rho_m y} R_a^{(0)}(dy) + \Delta^{(0)}(a)\delta_0(dy).$$

Remark 3 It can be seen from Proposition 20 that we come up with expressions included series for depletion quantities. These expressions might be not straightforward to compute but we can use numerical method to compute these expressions. We can also apply numerical methods to study other depletion quantities like $\mathbb{E}_x\left[e^{-q(\tau_a-\overline{G}_{\tau_a})}\right]$.

Fig. 3 Empirical probability of ruin before the first critical drawdown

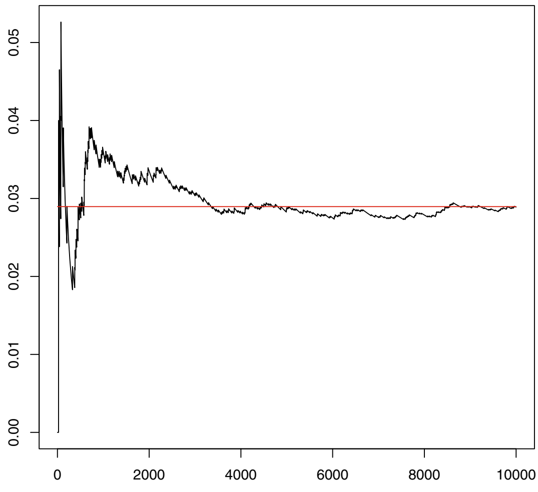
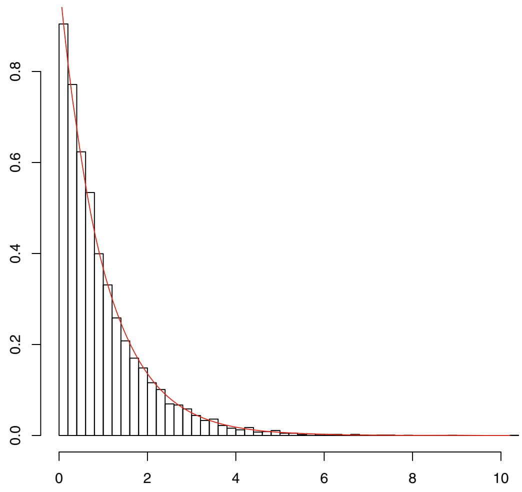


Fig. 4 Empirical distribution of the overshoot $Y_{\tau_a} - a$



6 Simulations

6.1 Classical Cramér–Lundberg mode with exponential claims

We consider a drifted compound Poisson process, introduced in Sect. 5.1, with $\lambda = 5$, exponential claims with parameter $\mu = 1$ starting from the initial capital $x > 0$ with the loading factor $\theta > 0$.

We first consider a sample of size $n = 10000$ of a such process on the time interval [0, 1000] and compute the depletion quantities by Monte-Carlo method with $a = 6$, initial capital $x = 5$ and loading factor $\theta = 1$. Note that $c = (1 + \theta)\lambda/\mu = 10$.

In Fig. 3, we observe the convergence of the empirical probability of ruin before the first critical drawdown (the black line) to its theoretical value (the red line), obtained by Formula (34), when n is increasing.

Figure 4 illustrates the second point of Proposition 8 by comparing the empirical distribution of the overshoot $Y_{\tau_a} - a$ over the critical drawdown and the exponential density with parameter μ (the red line).

In Fig. 5, we give the empirical distribution of τ_a, G_{τ_a} and the speed of depletion $\tau_a - G_{\tau_a}$ respectively and compare them to an exponential density (the red line) which seems to be a good estimation for the two first variables. We observe on the histogram of $\tau_a - G_{\tau_a}$ that with a positive probability τ_a is equal to G_{τ_a}.

We now observe, in Fig. 6, the evolution of the probability of ruin before the first critical drawdown in the loading factor θ and in the initial surplus x. The probability was computed using Formula 34.

As an illustration of the new potential of such quantities as risk management tools, we can now make a quick and interesting analysis from all of these graphs. We notice that the average claim is of one unit ($1) and that the initial level is set at five units $5. With an average premium rate of ten units ($10) we can now see that a

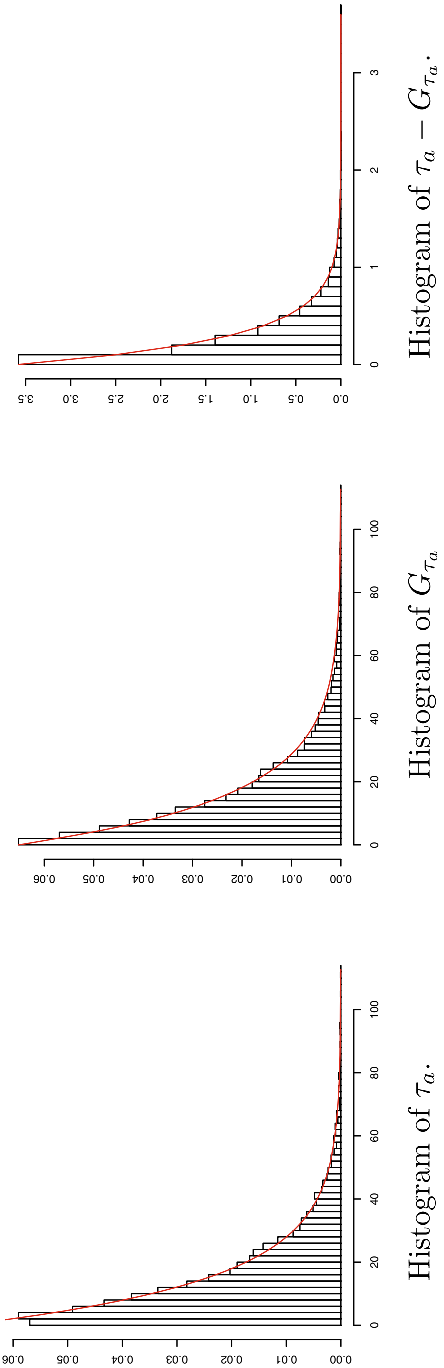
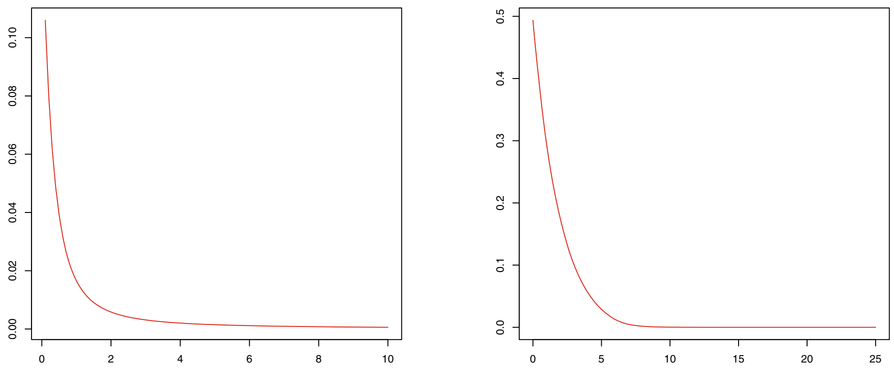


Fig. 5 Empirical distributions for a drifted compound Poisson process

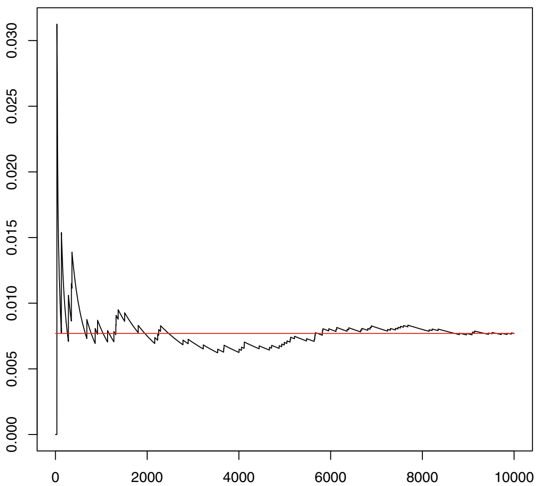


Evolution in $\theta \in (0, 10]$, with $x = 5$.

Evolution in $x \in [0, 25]$, with $\theta = 1$.

Fig. 6 Evolution of the probability of ruin before the first critical drawdown

Fig. 7 Empirical probability of ruin before the first critical drawdown



potentially dangerous drawdown of more than six units ($6) would occur most likely within 30 time units (lower left graph in Fig. 5) but when it occurs, it does rather fast; within only one time period (lower right graph in Fig. 5). Another useful piece of information is given in Fig. 4 where we can see that, when this drawdown occurs it is likely to overshoot the drawdown size of $6 by an amount somewhere between $1 and $3 units. That is a total drawdown of $9 in a worst case scenario happening in one single period.

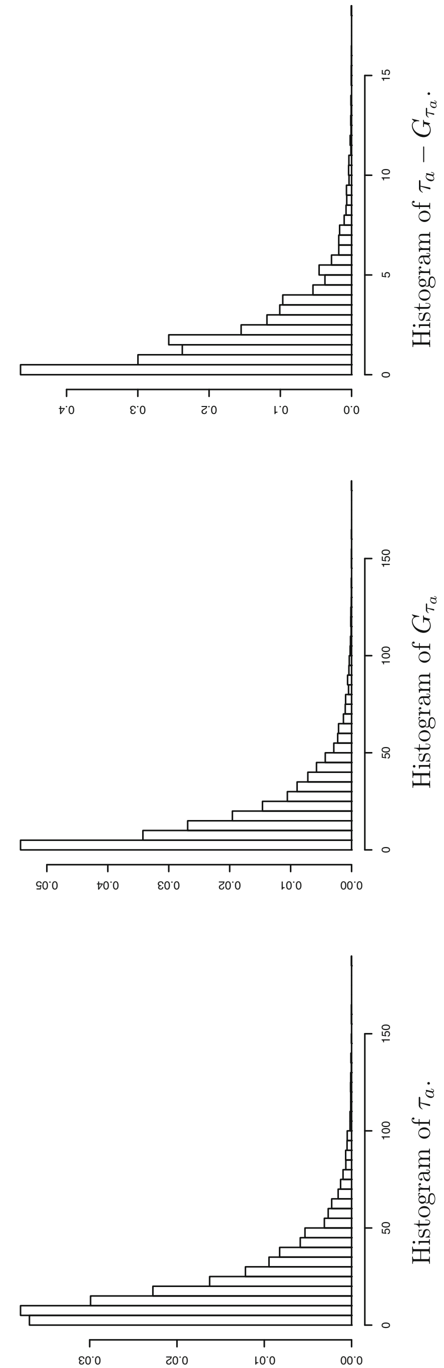


Fig. 8 Empirical distributions for a Gamma process

6.2 Gamma risk process

We now consider a Gamma risk process, introduced in Sect. 5.2, with $\alpha = \beta = 1$ starting from the initial capital $x = 5$ with the loading factor $\theta = 0.2$. The loaded premium is then equal to $c = (1 + \theta)\alpha/\beta = 1.2$.

We first consider a sample of size $n = 10000$ of a such process on the time interval $[0, T]$ with $T = 1000$ and compute the depletion quantities by Monte-Carlo method with $a = 3$.

We use a time step $h = T * 2^{-15}$ to generate the increments of a Gamma process on $[0, T]$.

In Fig. 7, we observe the convergence of the empirical probability of ruin before the first critical drawdown (the black line) to a positive value which is equal to 0.97% (the red line), when n is increasing.

In Fig. 8, we give the empirical distribution of τ_a, G_{τ_a} and the speed of depletion $\tau_a - G_{\tau_a}$ respectively. We observe that they are more distinct of an exponential density than in the Cramér–Lundberg model.

6.3 Spectrally negative stable process

We now consider a spectrally negative stable process, introduced in Sect. 5.3, with $\alpha = 1.2$ starting from the initial capital $x = 5$.

We first consider a sample of size $n = 10000$ of a such process on the time interval $[0, T]$ with $T = 2000$ and compute the depletion quantities by Monte-Carlo method with $a = 20$.

We use the algorithm introduced by Chambers et al., [5], and formalized by Weron in [23, 24], with a time step $h = 10^{-2}$, to generate the increments of a stable process on $[0, T]$.

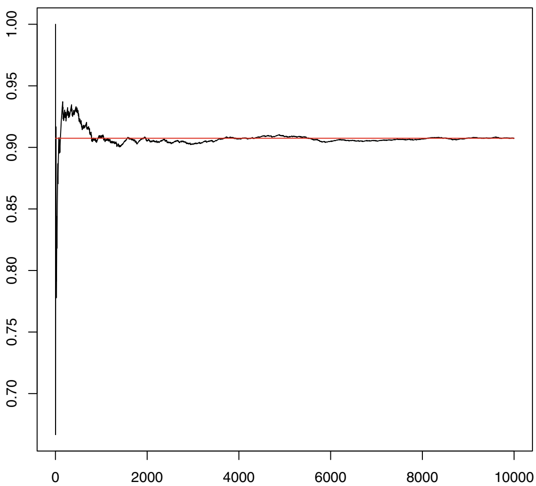


Fig. 9 Empirical probability of ruin before the first critical drawdown

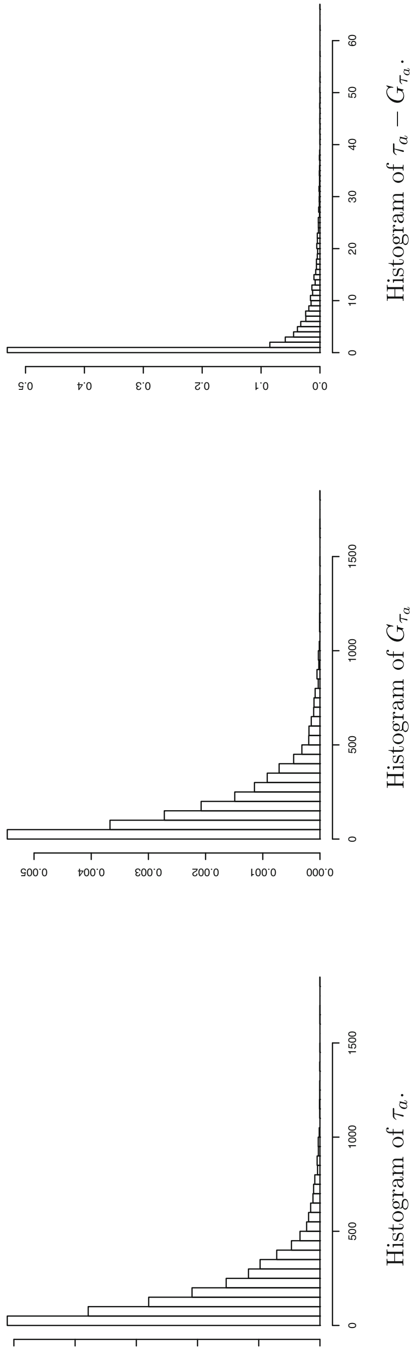


Fig. 10 Empirical distributions for a spectrally negative stable process

In Fig. 9, we observe the convergence of the empirical probability of ruin before the first critical drawdown (the black line) to a positive value which is equal to 0.405 (the red line), when n is increasing.

In Fig. 10, we give the empirical distribution for a spectrally negative stable Lévy process of τ_a, G_{τ_a} and the speed of depletion $\tau_a - G_{\tau_a}$ respectively. As in the Cramér-Lundberg model, it seems that with a positive probability τ_a is equal to G_{τ_a}.

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